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X-Rays Fluorescence & Compton Experiment, Variant Mass of the Electron at the Atom & Variant Mass for an Accelerated Charged Particle (Maxwell Radiation)

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ABSTRACT

The X Rays Fluorescence Experiment covers theory, x-rays detectors, measurement of the photon energy, data analysis, determination of the energy calibration, interpretation of the lines seen in the energy spectra, Moseley's Law, statistical methods, covariance ellipse and the maximum likelihood method. The Compton Experiment covers theory, Klein Nishina formula, Experiment, Calibration and data analysis, Measurements of the Energy of Compton Scattered Photons, Compton edge and the Compton Scattering Cross Section. Planck wrote in a letter about the research of the energy of the electrons at the cavity of the blackbody: "I knew that the problem of the equilibrium between matter and radiation is of high importance for the Physics. Besides, I knew the right formula for the distribution of the energy spectrum at the cavity. It was very important to find a theoretical interpretation". After 10 years of research, finally Planck accepted and proposed the quantum theory as explanation to the radiation of the blackbody. The introduction of the Planck constant h was fundamental for the quantization of the energy. The main argument to accept the quantum theory was the concordance with the entropy concept and the statistics thermodynamic. The Einstein research about the quantization of the light radiation at the Photoelectric Effect was also a support for the Planck research of the cavity of the blackbody. Millikan mentioned about the Photoelectric Effect: "The Photoelectric Effect is a proof independent of the quantum theory for the radiation of the blackbody as discrete and discontinue emission of energy absorbed by the electronic components of the atoms….". The Photoelectric Effect supports the constant h discovered by Planck at the research of the blackbody and that the Planck research corresponds to the reality". Bohr proposed a revolutionary concept for the behavior of the electron during the atom transition from one stationary level to other. In an article research, Bohr mentioned: "The correspondence principle has as consequence the comparison between the atom reaction due a radiation field with the reaction of the atom due a field from the mechanic classic point of view: which is due to a group of virtual harmonic oscillators with frequency equal to the determined for the possible transitions between stationary levels: hv=E-E´". The correspondence principle of Bohr postulated that for high quantum numbers the quantization approach is in correspondence with the classic theory. Therefore, Bohr postulated the quantization of the energy transition for the electrons at the atom (E-E´=hf) and the quantization of the angular momentum $L=n\frac{h}{2\pi}$. Bohr could explain the atom stability (the no radiation for the electrons at the atom) with those postulates and obtain a formula for the quantization of the energy, velocity, radius, angular momentum, frequency and wavelength of the radiation emitted or absorbed. Later, the modern quantum physics could explain the postulates of Bohr and obtain the quantization formula for the energy and angular momentum at the stationary levels by applying the Schröndiger Equation (wave probabilistic theory) and Heisenberg Theory (matrix theory). The stationary states or levels correspond to those functions which satisfy the Schröndiger Equation. The electron in an atom no excited is at rest. Thus, it cannot radiate energy because it corresponds to a stationary level of the atom.

For other hand, Albert Einstein wrote in a research article: "Does the inertia of a body depend on its energy content?" (Ist die Trägheit eines Körpers von seimen Energienhalt abhängig?): "If a body emits energy E in the form of radiation, its mass decreases by E/c^2". The fact that the energy that leaves from the body is converted into radiation energy makes no difference, so the more general conclusion is reached that the mass of a body is a measure of the content of its energy ... It is not impossible that with bodies whose content of energy is highly variable (for example radio salts) the theory can be successfully tested. If the theory corresponds to the fact, radiation conducts inertia between the bodies that emit and absorb it[1]". This is true for any type of radiation emitted (gravitational or electromagnetic energy) which produce a decrease in the mass of the body. Respect to the gravitational energy emission, it is demonstrated by theory, experiment and result the discovery formula which describe exactly the variant mass of a particle which emits gravitational energy which was demonstrated by myself at the article: The Fundament of the Mass and Effects of the Gravitation on a Particle and Light in the mass, time, distance, velocity, frequency, wavelength: Variant Mass for a Particle which emits Gravitational Energy for a particle orbiting a large Planet or Sun and for a Binary Star and Variant Frequency for the Light passing close a Gravitational Field from a Massive Object (Sun). The results of the mass formula are of great relevance for Gravitational Interactions. It is in accordance with the classic result for the emission of the total gravitational energy (bond total energy) for a particle orbiting a large Planet or Sun and for a Binary Star. At the atom, the electron only radiates this energy when it jumps from one orbit to another orbit at the atom. It is in accordance with the experimental results from the spectral lines of the atom. The difference is that in a gravitational field the particle or a planet around the sun can take any position at the space and any radius. But, the electron at the atom only can take restricted positions which are explained by quantum mechanics, and the electrons don´t emit radiation when they orbit around the nucleus.

In this article, it is demonstrated the development formula for the variant mass of the electron at the atom which describe exactly the variant mass of the electron (charged particle) at the atom which emits electromagnetic energy from one stationary level to other. The results of the formula are compared with the ionization energy emission for the electron at the atom and the bound energy for the diatomic molecules. The results are in agreement with high accuracy. Besides, Maxwell's theory shows that electromagnetic waves are radiated whenever charges accelerate as for example for the electrón. Then, this electromagnetic radiation (photons) produces a decrease in the mass of the electron which is given by the formula of the Variant Mass for an Accelerated Charged Particle which was demonstrated by me at this research. Therefore, an additional objective of this article is to demonstrate by theory, calculations and results the discovered formula which describe exactly the variant mass of an accelerated charged particle. This charged particle emits electromagnetic radiation which is called the Maxwell Radiation. Also, the article analyzes and establishes about the effects of the variant mass on the particle. In addition, the formula is tested with the electromagnetic radiation emitted for an electron when leaves from the atom. Finally, it is obtained the formula for the power energy emitted for an Accelerated Charged Particle.

Keywords: X-rays detectors, measurement of the photon energy, data analysis, determination of the energy calibration, interpretation of the lines seen in the energy spectra, Moseley's Law, statistical methods, covariance ellipse, the maximum likelihood method, Klein Nishina formula, Calibration, Measurements of the Energy of Compton Scattered Photons, Compton, the Compton Scattering Cross Section, X Rays Fluorescence, Compton Experiment, Atom, Variant Mass, Ionizing Energy Atom, Diatomic Molecules, Maxwell Theory, Einstein Mass-Energy, Accelerated Charged Particle, Electromagnetic Radiation, Ionizing Energy Atom, Power Energy.

1. X Rays Fluorescence Experiment

1.1. Interaction of photons with the matter

The interaction of photons with the matter is characterized by the fact of that each x ray photon is removed individually from the incident beam in a single event [1],[2]. Thus, the number of photons removed ΔB, is proportional to the traversed thickness Δx, and to the number of incident photons B:

$\Delta B = -\mu B \Delta x$ (μ: proportionality constant called the attenuation coefficient)

The absorption occurs if only a certain fraction of the radiation (B_o) passes through the absorber. If it happens, the wavelength of the transmitted beam is unchanged and the transmitted beam B (if the radiation is homogeneous, μ is a constant) is described by the next formula (which is called attenuation of the radiation in matter) [45]:

$B = B_o e^{-\mu x}$

A number of photons equal to (B_o-B) have been lost in the absorption process, most of this loss being due to the photoelectric effect. The value of the μ constant referred in the above equation is a function of both: the photoelectric absorption (τ) and the scatter radiation (σ): $\mu = f(\tau) + f(\sigma)$

However, $f(\tau)$ is usually large in comparison with $f(\sigma)$.

Because the photoelectric absorption is made up of the absorption in the various atomic levels, it is a dependent function of the atomic number. A plot of μ/ρ against λ or energy E contains a number of discontinuities called absorption edges, at wavelengths corresponding to the binding energies of the electrons in the various subshells [1]-[4],[45].

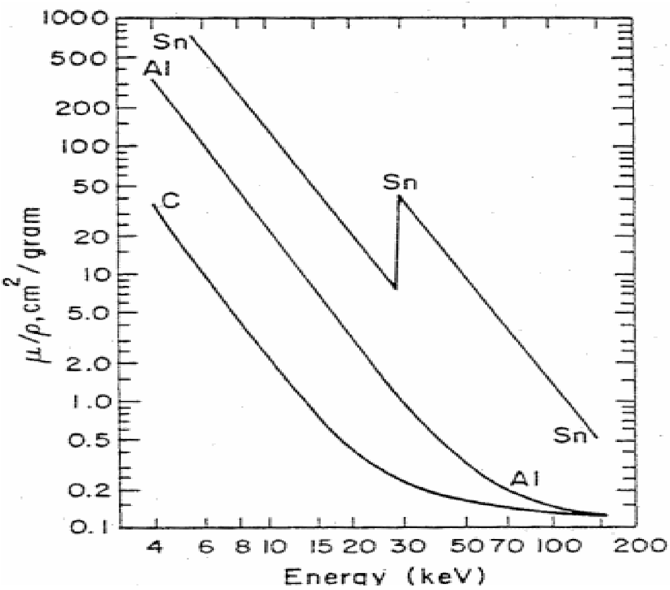

Fig.1. Mass attenuation coefficient (μ/ρ) as a function of the energy for several elements [45]

In the energy range from about 50 keV to about 50 MeV, the most of the interactions are due to:

a) Coherent (Rayleigh) effect

This effect occurs only with a low energy of the incident photon. We have only scattering of the incident photon and no ionization. The x-ray photon collides with one of the electrons of the absorbing element and no energy is lost in the collision process (so, the scattered radiation will retain exactly the same wavelength as the incident beam) [1], [2], [3], [4], [45]. The probability of this effect is proportional to $\frac{Z^{2.6}}{E^2}$.

b) Photoelectric Effect (Quantization of the radiation: light)

For the energy range in the X-Rays experiment the photoelectric effect is dominant. It occurs for low energy photons E<200 keV. This effect can be explained at the microscopic level by means of the electromagnetic radiations as composed of photons. Thus, an incident photon with sufficient energy is completely absorbed by a bound electron of the atom in the absorber material, and the electron is ejected (photoelectron). Thus, the photon gives all its energy to a bound electron (recoil electron) of the atom (bound electron is required in order to conserve the energy and the momentum), which uses part of the energy to overcome its binding to the atom and takes the rest as kinetic energy. The vacancy left in the atomic structure by the ejected electron is filled by one of the electrons from a higher shell or orbit or by a free electron from outside of the atom [1]-[4].

Then, the transition of the electron from a higher shell is accompanied by an emission of a x-ray or this x-ray emitted can be imparted to another electron, which is emitted and it is called Auger electron. The subject of this study is called Fluorescence. The interaction is again dependent of Z, and an approximate expression for the absorption probability τ is proportional to $\frac{Z^n}{E^{3.5}}$ where, n is normally between 4 and 5 depending of the absorbing material. The Photoelectric Effect can also be explained at the wave theory by the next form: the electromagnetic radiation incident on the surface which consists of electric and magnetic fields can exert forces (mainly the electric field) on the electrons of the surface and therefore, they can be emitted [1]-[4], [45].

It was observed at the Photoelectric effect that if the frequency of the incident radiation is very low, then there is no electron ejected. The classic electrodynamics has established that the electrons must be emitted in any frequency if the intensity is enough. The quantization of the electromagnetic radiation solved the contradiction from classic electrodynamics for this experiment [5]. Einstein proposed a corpuscular theory for the incident radiation (light) at the Photoelectric Effect [5]. The formula of the Photoelectric effect is as follows: $hf=K_{max}+U_o$ where U_o is the bound energy of the electron, hf is the electron energy and K_{max} is the maximum kinetic energy of the electron [1], [2], [3], [4], [45]. The bound energy U_o was called work function. It is equal to the minimum energy $E=hf_{min}$ necessary to eject an electron with kinetic energy almost with zero value: $hf_{min}=U_o$ $f_{min}=U_o/h$ U_o=work function K=0 for f_{min}.

If $f<U_o/h$ (very low frequency, very low energy), then there is no electron ejected independent of the radiation intensity. If the frequency is increased until the value where the energy of a single photon E=hf is more than the bound energy of the electron or work function U_o: $hf\geq U_o$, $f\geq U_o/h$, and then the electron is ejected. The rest of the

energy is given to the electron as kinetic energy. Thus, the electron has absorbed the energy of a single photon (with energy E=hf) from the electromagnetic radiation [4], [5]. If the radiation intensity (with minimum frequency $f>U_o/h$) increases, then more electrons can be ejected.

The maximum kinetic energy of the electron increases linearly with the frequency: $K_{max}=hf-U_o$. Then, it depends on the energy of the incident radiation which at the microscopic level is composed of quantum packets whose energy depends of the frequency (E=hf). It is concluded that for a fixed value of frequency, the maximum kinetic energy of the electron always is the same independent of the radiation intensity. Nevertheless, the number of electrons emitted increases with the intensity. Besides, some electrons can be ejected with a kinetic energy less than the K_{max} because some electrons can lose energy in the collision process at the atom [4], [5].

Planck proposed a quantization of the energy for the electron: virtual oscillators in the cavity walls of the blackbody (wave theory). Einstein proposed a corpuscular theory for the incident radiation (light) [5]. This photoelectric effect was a proof of the analogy of the atom with the Blackbody Research of Planck: the electrons at the cavity of the blackbody behave as virtual oscillators which absorb and emit energy in discrete packets of energy or photons. At the Photoelectric Effect, the energy of the incident radiation is quantitated as discrete quantum energy called photons [1], [2], [3], [4], [45]. In resume, this Photoelectric Effect was a proof that the energy of the incident radiation (electromagnetic radiation) is composed of packed of energy called photons with energy E=hf which is given to the bound electron. It was also a proof of stationary levels of energy at the atom and the quantization at the Blackbody research from Planck. Therefore, at the microscopic level, there is emission or absorption of energy as discrete packets of energy called photons where the incident energy radiation is the sum of all discrete packets of energy. In 1916, Millikan conducted a series of experiments that confirmed Einstein's theory of the photoelectric effect. This experiment demonstrated the quantization of the light radiation and the corpuscular theory of Einstein [5], [45]. It was also possible to determine Planck's constant with this experiment [5], [45].

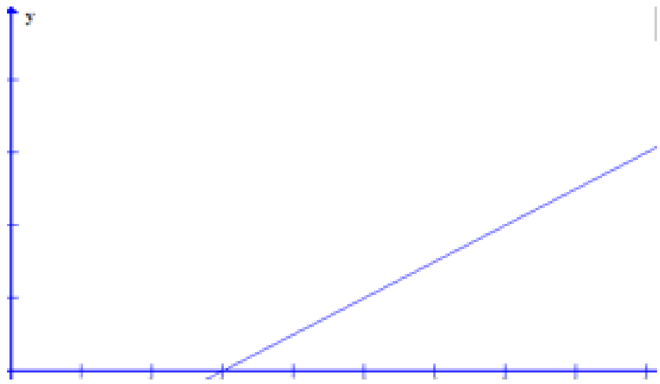

Fig.2. Graphic of the maximum kinetic energy of the photoelectrons versus the frequency of the light for the Photoelectric Effect

The figure shows a linear relationship between the maximum kinetic energy and the frequency: $K_{max}=hf-U_o$. The constant of Planck was possible to obtain from the slope of the graphic of the maximum electrical potential to stop the electron flux versus the frequency:

$hf=K_{max}+U_o \quad K_{max}=(1/2)\,mv^2=eV_{max}$

V_{max}: electrical potential necessary to stop the electron flux: retardation potential.

$hf= eV_{max} +U_o \quad V_{max}=(h/e)f-(U_o/e)$

$s=h/e$ (s: slope)

$h=se$ where s is the slope and e is the electric charge of the electron

This same constant that is used for the quantization of the virtual electron oscillators in the black body cavity is used to quantize electromagnetic radiation in the Photoelectric Effect. This constant is called the Planck's constant [5], [45].

It is also possible to determine U_o when $V_{max}=0$: $U_o=hf_{min}$

c) Compton scattering (Incoherent scattering)

At energies much greater than the binding energies of the electrons, the photons are scattered as if the electrons were free and at rest. In this case, an incident photon scatters from an outer shell electron in the absorber material at an angle Φ, and part of the photon energy is imparted to the electron as kinetic energy. Typically, the photon will have sufficient energy to produce several recoil electrons. Then, in a x-ray beam all possible energy losses will occur. The net result is the production of ion cascades as fast electrons react with other atoms. The photon never lost the whole energy in any one of the collisions. The scattered photons can then continue through the absorber and interact again or scatter out of the absorber material completely. It is the dominant mode of interaction around 1 MeV (intermediate energy range) [1], [2], [3], [4], [45]. The probability decreases rapidly with the increasing energy and it is also dependent on the number of electrons available for the scattering of the photon, and hence it increases proportionally with the increasing atomic number Z. Thus, the probability of this effect is proportional to Z/E. If the full energy of the incident photon is not absorbed in the detector, then there is a continuous background in the energy spectrum, known as the Compton Continuum. It extends up to an energy corresponding to the maximum energy transfer, where there is a sharp cut-off point, known as the Compton Edge [1], [2], [3], [4], [45].

d) Pair Production

Pair production (high energy γ rays (photons) from 5 to 10 MeV) is the process in which a photon in the field of a nucleus or an electron disappears with the creation of an electron-positron pair. The total kinetic energy of the resultant particles is equal to the photon energy minus the mass energy of the two particles which have been created. The cross section for this process shows that it varies with Z approximately as Z^2 and increases with increasing photon energy. For materials with higher atomic number Z, this dominance will occur at lower energies. If the energy of the incident photon is greater than 1.022 MeV (twice the electron rest mass) in the presence of an atomic nucleus, then an electron/positron pair can be produced. Any residual energy is distributed evenly between the electron and the positron as kinetic energy. Afterwards, the positron is annihilated with one of the atomic electrons producing two γ rays of energy 511 keV [1], [2], [3], [4], [45].

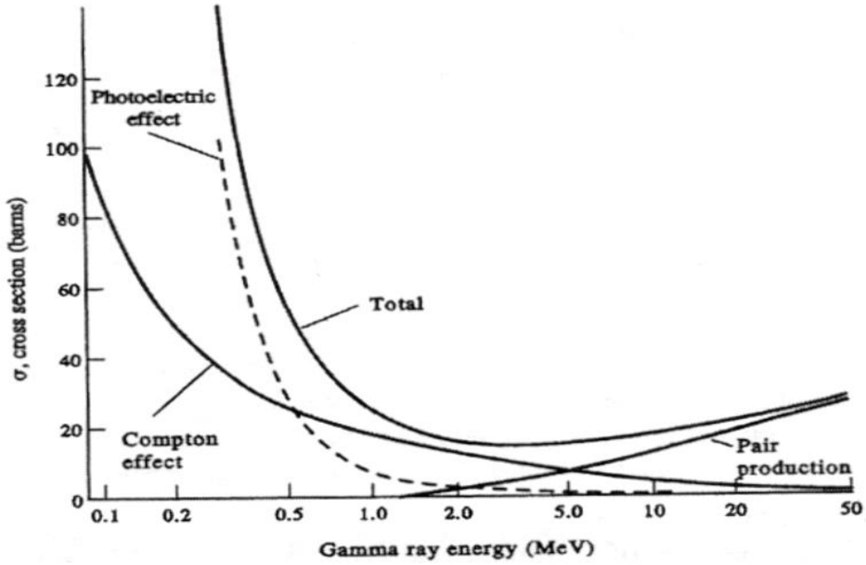

Fig.3. Relative probability (given by σ: cross section) of each type of interactions as a function of the energy in lead [45]

1.2. X-Rays Theory and X-Rays Fluorescence process

The X-rays can be described in 2 forms: electromagnetic wave or particle (photon). The photon energies of the X-rays (in the electromagnetic spectrum) are between 100 eV and 100 keV, the wavelengths between 10^{-8} m and 10^{-12} m, and the frequencies between 10^{16} Hz and 10^{20} Hz. The X-rays were discovered by Wilhelm K. Roentgen in 1895 and in 1913 Henry Moseley measured and plotted the x-rays frequencies for about 40 of the elements of the periodic table. Moseley showed that the K_α X-rays followed a straight line when the atomic number Z versus the square root of the frequency v was plotted [5], [8], [9], [45]. We can write his empirical relationship as follows:

$$hv_{K\alpha} = 13.6\text{ eV }(Z-1)^2\left(\frac{1}{1^2}-\frac{1}{2^2}\right) = \frac{3}{4}13.6(Z-1)^2\text{eV}$$

$$Z = \sqrt{\frac{4hv_{K\alpha}}{3(13.6)*1{,}6*10^{-19}}} + 1$$ (atomic number versus frequency v in Hz)

$$Z=\sqrt{\frac{v_{K\alpha}}{0.246*10^{16}}} + 1=2.02\sqrt{\frac{v_{K\alpha}}{10^{16}}} + 1$$

$$Z=2{,}016*10^{-8}\sqrt{v_{K\alpha}} + 1$$

With the insights from the Bohr model, we can write this empirical relationship as follows: $\Delta E = hv = hcRZ^2(\frac{1}{n'^2}-\frac{1}{n^2})$ **energy spectral lines**

$$\Delta E = hcR(Z-1)^2\left(\frac{1}{n'^2}-\frac{1}{n^2}\right) \qquad R=\frac{me^4}{8\varepsilon_o{}^2ch^3}$$: Rydberg Constant

hcR: Rydberg in unit of energy=13.6 eV

(Z- σ)=(Z-1) σ=1: shielding constant (σ≤1) n′=1 n=2: Kα lines

$$\Delta E = hv_{K\alpha} = 13.6\ \ (Z-1)^2\left(\frac{1}{1^2}-\frac{1}{2^2}\right)\text{eV} = \frac{3}{4}13.6\text{ eV }(Z-1)^2\text{eV}$$

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$$Z = \sqrt{\frac{4h}{(3)(13{,}6)}}\sqrt{v_{K\alpha}} + 1 \qquad Z = 2{,}016 * 10^{-8}\sqrt{v_{K\alpha}} + 1$$

Z versus $\sqrt{v_{K\alpha}}$: Moseley Plot

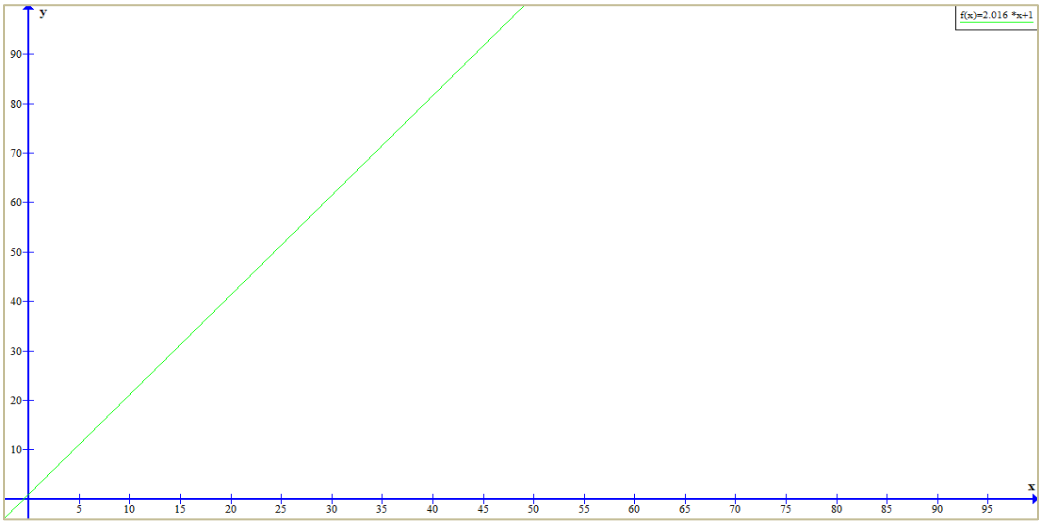

Fig.4. Moseley's plot: atomic number Z vs square root of frequency $\sqrt{v_{K\alpha}} * 10^{-8}$ (v in Hz) for the K_α lines

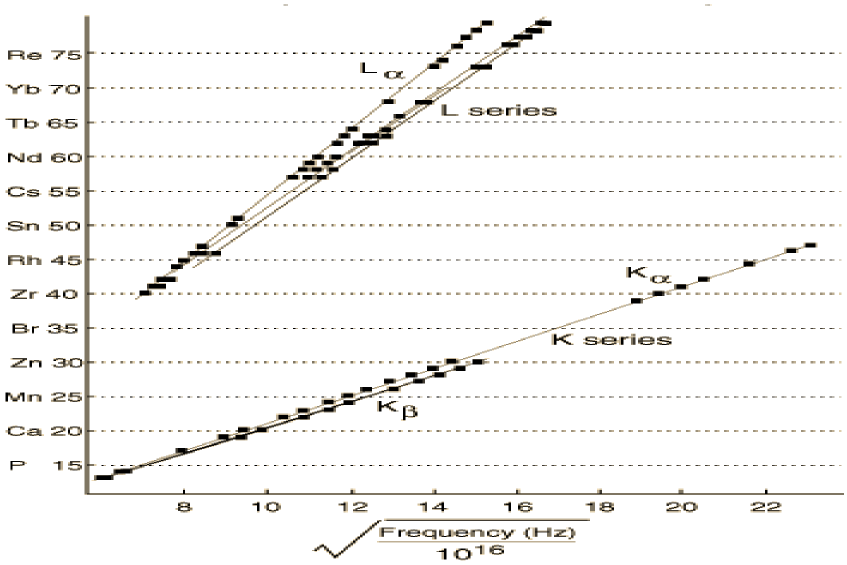

Fig.5. Moseley's plot: the atomic number vs square root of the frequency $(\frac{v}{10^{16}})$ v in Hz [45]

For other hand, the X-rays fluorescence (XRF) is a process where a material is exposed to X-rays of high energy, and as the X-rays (or photons) strike an atom (or a molecule) in the sample, the energy is absorbed by the atom. If the energy is high enough, a core electron is ejected out of its atomic orbital. Then, an electron from an outer shell drops into the unoccupied orbital to fill the hole left behind. This transition of the electron gives off a X-ray of fixed characteristic energy (X-ray fluorescence) which can be detected by a fluorescence detector. The energy needed to eject a core electron is characteristic of each element and also the emitted energy by the transition [5], [8], [9], [45]. Therefore, it is possible to know what elements a material is made through the fluorescence of X-rays and by a fluorescence detector.

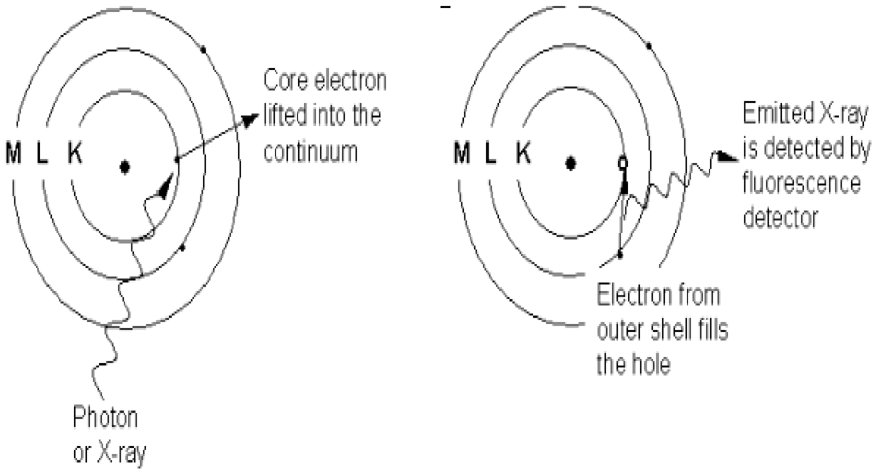

Fig.6. The X-ray fluorescence process [45]

1.3. Sources of X-rays

X rays Tube

One form of generating X rays is the stopping of energetic electrons in the electromagnetic field of the atomic nuclei. A high vacuum tube is usually used for it. The unit of primary source consists of a very stable high-voltage generator, capable of providing a potential of typically 40–100 kV. The current from the generator is fed to the filament of the x-rays tube. X-rays are produced when the electrons are suddenly decelerated with a high voltage and directed to the collision with the metal target (anode). The X rays are produced due the fact that accelerated charges give off electromagnetic radiation as Maxwell Theory establishes.

These X-rays are commonly called bremsstrahlung or "braking radiation“ (white radiation also). Therefore, part of the kinetic energy of the electron is transformed into X-rays (bremsstrahlung) but the bigger part of this energy is transformed into thermal energy of the anode. Then, it must be very heat resistant. Besides, if the bombarding electrons have sufficient energy, they can knock and put out an electron of an inner shell of the atoms of the metal target. Then, the electrons from the higher states drop down to the vacant shell, emitting characteristic (fluorescence) x-rays (photons) with precise energies determined by the energy levels of the electrons.

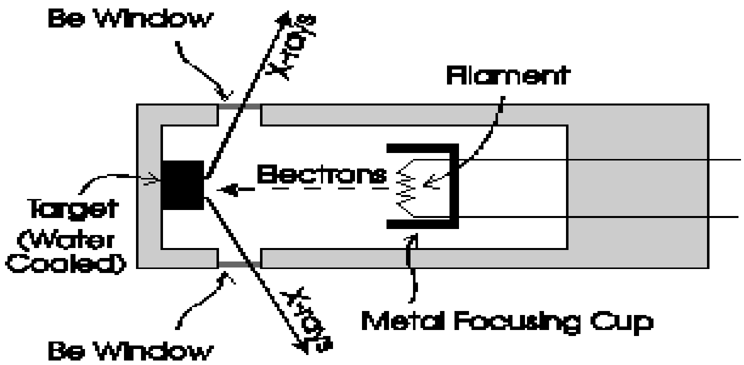

Fig.7. Scheme of a x-rays tube [45]

The minimum wavelength λ (in pm), can be derived from the energy E (in keV) o V(in kV), by the relationship: λ_{min}=1239.81/E or λ_{min}=1239.81/V. Ulrey bombarded tungsten targets with electrons of four different energies and obtained the graph of the intensity against the wavelength of the emitted X-rays (for some accelerating voltages) [5], [8], [9], [45].

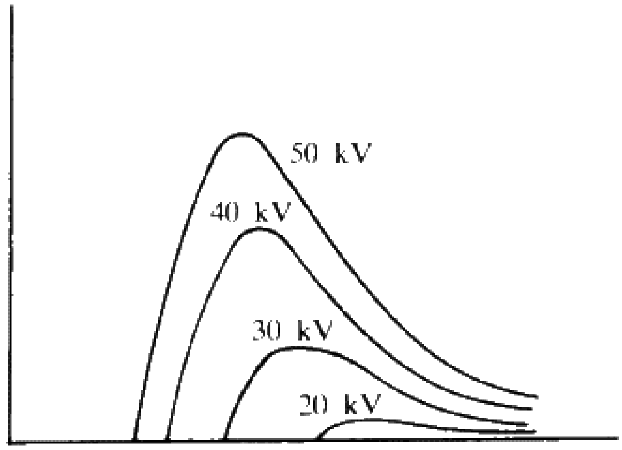

Fig.8. Intensity (y abscissa) versus the wavelength of the X-rays (x abscissa) [45]

The efficiency is low and the intensity grows with the atomic number Z of the anode. The maximum intensity is at 1.5 λ_{min}. It is characterized by a continuous distribution of radiation which becomes more intense and shifts toward lower wavelength when the energy of the bombarding electrons is decreased or shifts toward higher frequencies when the energy of the bombarding electrons is increased. The general fall of the x-rays absorption coefficient with the increasing energy of the incident photon is interrupted by a sharp rise when the energy is equal to the binding energy of an electron shell (K, L, M, etc.) of the absorber material. This energy is the minimum energy at which a vacancy can be created in the particular shell and it is referred to as the ‘edge’ or ‘critical excitation’ energy. They are generated when an ‘initial’ vacancy in an inner shell (created by a x-ray or an electron excitation), is filled by another electron transfer from other shell, thus leaving a ‘final’ vacancy in that shell. The energy of the line is equal to the difference in the binding energies of the shells of the ‘initial’ and ‘final’ vacancies. Depending on the atomic number, the x-ray spectra from the elements can include lines from the K, L, M, N and O series corresponding to the excitation of the K, L, M, N or O levels. Lines are identified both by the common labels as for example: $K_{\alpha 1}$, $K_{\alpha 2}$, etc., or the term labels with the ‘initial’ and ‘final’ vacancies as for example: KLIII, KLII, etc. The relationship between the wavelength of a characteristic x-ray photon and the atomic number Z of the excited element was firstly established by Moseley [5], [8], [9], [45]. With the insights from the Bohr model, it is as follows:

$h\nu_{K\alpha} = \frac{3}{4}hcR(Z-1)^2 eV$ (hcR: Rydberg in unit of energy: 13.6 eV)

$\frac{hc}{\lambda} = \frac{3}{4}hcR(Z-1)^2 eV$

$\frac{1}{\lambda} = \frac{3}{4}R(Z-1)^2 eV$ (R=$\frac{me^4}{8\varepsilon_o{}^2ch^3}$: Rydberg Constant)

(Z- σ)=(Z-1) σ=1: shielding constant σ≤1 n′=1 n=2: Kα lines

$\delta = R(\frac{1}{n'^2} - \frac{1}{n^2})$ $\delta = \frac{3}{4}R$ for the K_α lines

$\frac{1}{\lambda} = \delta(Z - \sigma)^2 \quad \frac{1}{\sqrt{\lambda}} = \sqrt{\delta}\,(Z\text{-}\sigma)$

Where, δ is a constant which takes different values for each spectral series, σ is the shielding constant which has a value of just less than unity. In the case of a metal with a small atomic number such as copper or molybdenum, we observe very characteristic lines.

The characteristic lines are caused by electrons being knocked out of the K shell of an atom and then the electrons from the L shell fill the vacancies in this K shell. The emitted energies in this process corresponds to the so-called K_α (n=2 to n=1) and K_β (n=3 to n=1) lines. For other hand, the transitions to the n=2 or L shell are designated as L x rays. For example, n=3 to n=2 is L_α, n=4 to n=2 is L_β, etc.

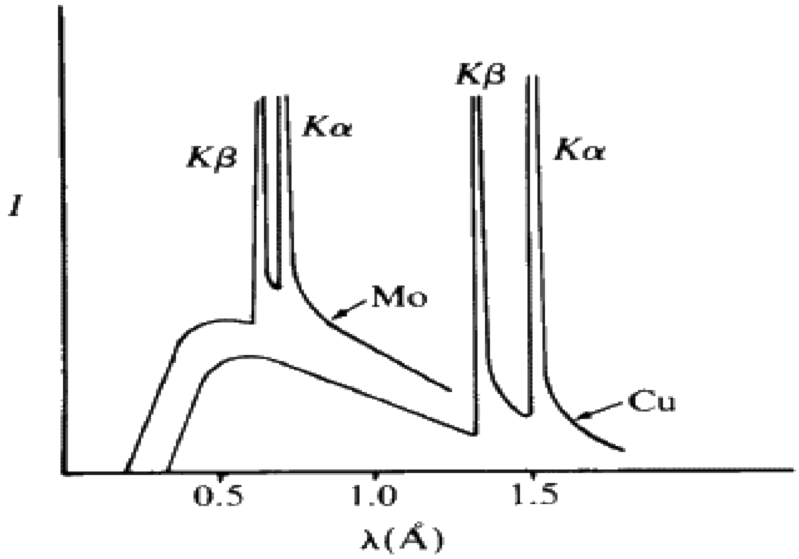

Fig.9. Characteristic x ray peaks from a x rays tube anode (Cu and Mo): Intensity versus wavelength of X-rays [45]

The continuous distribution of x rays which forms the base for the two sharp peaks is the bremsstrahlung radiation (braking radiation or white radiation). The probability that a vacancy in a given shell will result in the emission of a X-ray is the fluorescence yield of this shell. Not all the vacancies result in the production of characteristic X-ray photons since there is a competing internal rearrangement process known as the Auger effect. The ratio of the vacancies resulting in the production of characteristic X-ray photons to the total number of vacancies created in the excitation process is called the fluorescent yield [5], [8], [9], [45]. The selection rules for the production of normal lines require that the principal quantum number n must change by at least one, the angular quantum number l must change by +/- 1, and the J quantum number (where J= l+ s and s is the spin quantum number) must change by 0 or 1. In effect, it means that for the K series only p,s transitions are allowed, yielding two lines for each principal level change. Vacancies in the L level follow similar rules and give rise to L series lines. There are more of the L lines since there are more allowed transitions. In practice, the number of observed lines of a given element will depend of the atomic number of the element, the excitation conditions and the wavelength range of the employed spectrometer [5], [8], [9], [45]. While the most of the observed fluorescent lines are normal, certain lines may also occur in the X-ray spectra that do not fit the basic selection rules which are called forbidden lines and they are shown in the center portion of the below figure.

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The X-ray spectra is the plot of the reciprocal of the square root of the wavelength as a function of the atomic number, for the K, L and M series:

$\frac{1}{\sqrt{\lambda}} = \sqrt{\delta}\,(Z\text{-}\sigma)$ $\delta = R(\frac{1}{n'^2} - \frac{1}{n^2})$ $\delta = 3R/4$ for the K_α lines

A scale in wavelength is also shown to indicate the range of wavelengths over which a given series occurs.

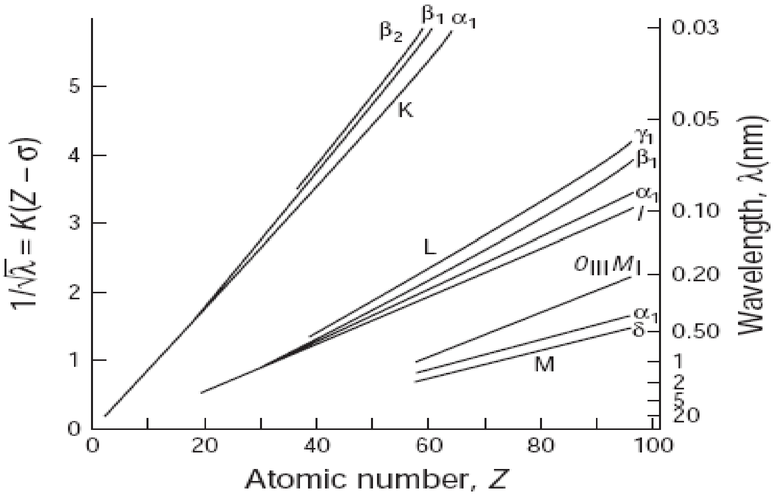

Fig.10. Moseley's plot indicating normal and forbidden fluorescent lines: plot of $\frac{1}{\sqrt{\lambda}}$ versus Z where $\frac{1}{\sqrt{\lambda}} = \sqrt{\delta}$ (Z-σ) and K=$\sqrt{\delta}$ [45]

Radioactivity

It refers to the particles which are emitted from the nucleus as a result of the nuclear instability. Because the nucleus supports the intense conflict between the two strongest forces in the nature (strong and electromagnetic forces), it should not be surprising that there are many nuclear isotopes which are unstable and emit some types of radiation.

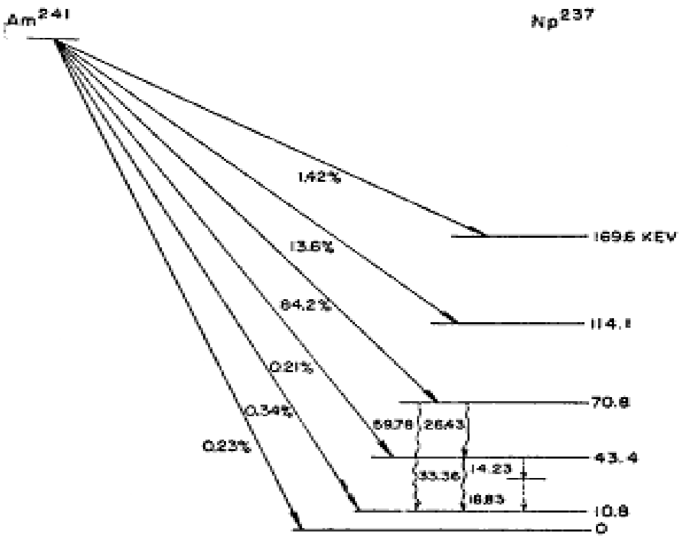

Fig.11. Decay of Am (241) in Np (237) [45]

The most common types of radiation are called alpha, beta, and gamma radiation, but there are other varieties of radioactive decay. After a radioactive decay, the daughter-nucleus is often in an excited state and it can get into the ground-state by emitting a gamma-photon. These photons can only have discrete energy values. The different types of radioactivity bring to different decay paths which transmute the nucleus into other chemical elements [5], [8], [9], [45]. It is possible to get monoenergetic X-rays photons with the decay product of Am (241) in Np (237) as it is shown in the next figure:

1.4. X-rays Detector (transducer for converting x-rays photon energy into voltage pulse)

The detectors work through a process of photoionization in which the interaction between the incoming X-ray photon and the active material (detector) produces a number of electrons. The current produced by these electrons is converted to a voltage pulse by a capacitor and a resistor, such that one digital voltage pulse is produced for each incoming X-ray photon. In addition, the detector must be sensitive to the appropriate photon energies, which means that it must be applicable to a given range of wavelengths or energies.

There are two other important properties that an ideal detector should possess: proportionality and linearity. Each incoming x-ray photon which enters to the detector produces a voltage pulse, and if the size (amplitude) of the voltage pulse is proportional to the photon energy, then the detector is called proportional. It is needed when the technique of pulse height selection is used. Pulse height selection means rejecting pulses of voltage levels (by using electronics) other than those pulses corresponding to the characteristic lines being measured. For other hand, X-rays enter to the detector at a certain rate and if the output pulses are produced at the same rate, the detector is called linear. Linearity is very important when the various count rates produced by the detector are used as measures of the photon intensity for each measured line [5], [8], [9], [45].

1.5. The proportional gas detector, the avalanche formation

The proportional gas detectors are almost always operated in the pulse mode. It relies on the phenomenon of gas multiplication to amplify the charge represented by the original ion pairs created within the gas. Gas multiplication is a consequence of increasing the electric field within the gas to a sufficient high value. At low values of the field, the electrons and ions created by the incident radiation simply drift to their respective collecting electrodes. During the migration of these charges, many collisions occur with neutral gas molecules. Because of their low mobility, positive or negative ions achieve a very little average energy through the collisions [5], [8], [9], [45].

Free electrons are easily accelerated by the applied field and they have significant kinetic energy when undergoing to this collision. If this energy is greater than the ionization energy of the neutral gas molecule, it is possible to create an additional ion pair in the collision (secondary ionization). Because the average energy of the electron through the collisions increases with the increasing electric field, there is a threshold value of the field above which this secondary ionization will occur. In typical gases and at atmospheric pressure, the threshold field is of the order of 10^6 V/m. Also, the electron liberated by this secondary ionization process will be accelerated by the electric field. During its subsequent drift, it undergoes to collisions with other neutral gas molecules and it can create additional ionization [5], [8], [9], [45].

Therefore, the gas multiplication process takes the form of a cascade, known as the Townsend avalanche, in which each free electron created in the collision can potentially create more free electrons by the same process [5], [8], [9], [45]. The fractional increase in the number of electrons per unit path length is governed by the Townsend equation: $\frac{dn}{n} = \alpha dx.$

In this equation, α is called the first Townsend coefficient for the gas. Its value is zero for the electric field values below the threshold and generally increases with the increasing field strength. For a spatially constant field, α is a constant in the Townsend equation. The solution of the differential equation establishes that the density of electrons grows exponentially with the distance as the avalanche progresses: $n(x)=n(0)e^{\alpha x}$.

For a cylindrical geometry used in most of the proportional counters, the electric field increases in the direction that the avalanche progress. In the proportional counter, the avalanche finishes when all free electrons have been collected at the anode. Under proper conditions, the number of secondary ionization events can be kept proportional to the number of primary ion pairs formed and the total number of ions can be multiplied by a factor of many thousands. The formation of an avalanche involves many energetic collisions electron-atom in which a variety of excited atomic (or molecular) states may be formed [5], [8], [9], [45].

1.6. Regions of the Detector Operation

At very low values of the voltage, the field is insufficient to prevent recombination of the original ion pairs, and the collected charge is less than the original collected charge by the original ion pairs. As the voltage is raised, the recombination is suppressed and the region of ion saturation is achieved. This mode is the normal operation mode for ionization chambers. If the voltage is still increased, the threshold field at which the gas multiplication begins is reached. Then, the collected charge begins to multiply, and the pulse amplitude will increase [5], [8], [9], [45].

In some regions of the electric field, the gas multiplication will be linear, and the collected charge will be proportional to the number of original ion pairs created by the incident radiation. It is the region of proportionality and linearity and represents the operation mode of conventional proportional counters and we use the detector in this region [5], [8], [9], [45]. If we increase the applied voltage or the electric field still more, it can introduce nonlinear effects. The most important is related to the positive ions, which are also created in each secondary ionization process. Although the free electrons are quickly collected, the positive ions move much more slowly and during the time that it takes to collect the electrons, they barely move at all.

Thus, each pulse within the counter creates a cloud of positive ions, which are very slow to be dispersed as they drifts toward the cathode [5], [8], [9], [45]. If the concentration of these ions is sufficiently high, they represent a space charge that can significantly change the shape of the electric field within the detector. Because further gas multiplication is dependent on the magnitude of the electric field, some nonlinearities will begin to be observed. These effects mark the onset of the region of limited proportionality and linearity in which the pulse amplitude still increases with the increasing number of initial ion pairs, but not in a linear form. If the applied voltage is made sufficiently high, the space charge created by the positive ions can become completely dominant to determine the next history of the pulse [5], [8], [9], [45].

Therefore, the avalanche proceeds until a sufficient number of positive ions have been created to reduce the electric field below the point at which an additional gas multiplication can take place [5], [8], [9], [45].

Then, the process is self-limiting and it will finish when the same total number of positive ions have been formed regardless of the number of the initial ion pairs created by the incident radiation. Then, each output pulse from the detector is of the same amplitude and it does not more reflects any property of the incident radiation [5], [8], [9], [45]. This region is the Geiger Müller region of operation, with lower counting rates, higher voltage and no proportionality. The regions of the detector operation are shown in the next graph:

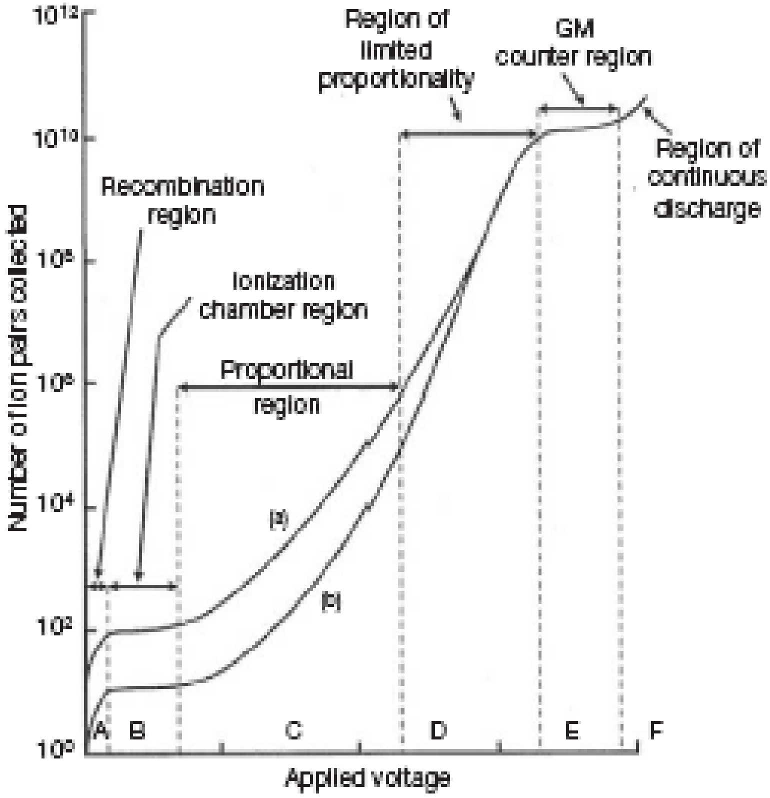

Fig.12. Regions of the detector operation [45]

1.7. Geometry of the detector

The anode consists of a fine wire which is positioned along the axis of a large hollow tube that serves as the cathode. The polarity of the voltage is important because the electrons must be attracted toward the center axial wire [5], [8], [9], [45]. This voltage is essential for two reasons:

1.- Gas multiplication requires large values of the electric field.

It is given by: $E(r) = \frac{V}{r} \ln(\frac{b}{a})$

Where, V is the voltage applied between the anode and the cathode, a is the anode wire radius, b is the cathode inner radius.

Therefore, large values of the electric field occur in the immediate vicinity of the anode wire where r is small. As the electrons are attracted to the anode, they are attracted toward the high field region.

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2.- If uniform multiplication is achieved for all ion pairs formed by the original radiation interaction, the region of gas multiplication is confined to a very small volume compared with the total volume of the gas. Under these conditions, almost all primary ion pairs are formed outside the multiplication region, and the primary electron simply drifts to that region before the multiplication takes place. Therefore, each electron undergoes the same multiplication regardless of its original position of formation, and the multiplication factor wil be the same for all original ion pairs [5], [8], [9], [45].

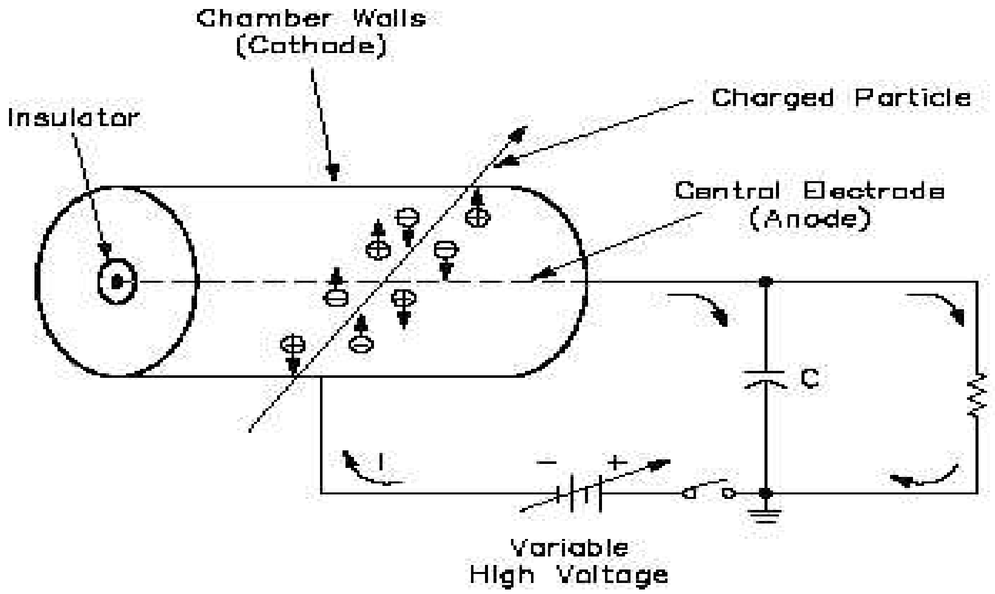

Fig.13. Cylindrical geometry for the proportional counter [45]

1.8. Absorption in the used detectors

We can see in the next figure the absorption efficiency versus the energy for different gases at normal pressure:

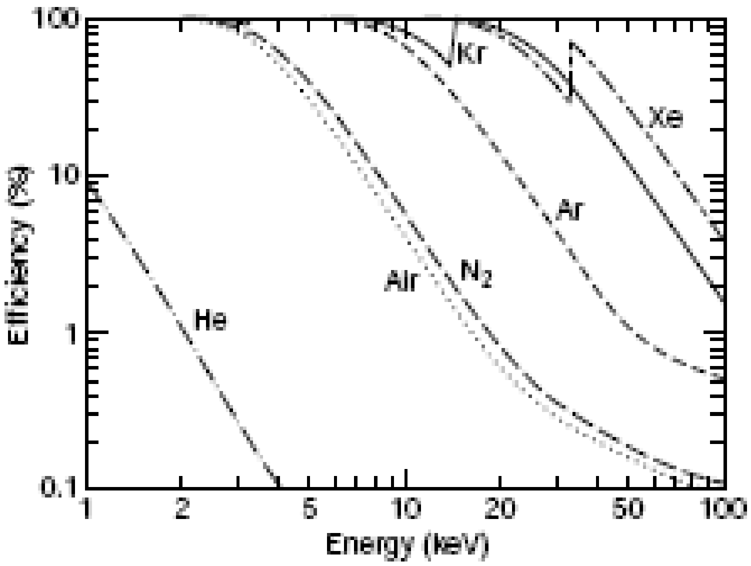

Fig.14. Absorption efficiency vs energy for He, Air, N_2, Ar, Kr and Xe [45]

We use in this experiment as detectors two proportional counters filled with Argon and Xenon. As a quench gas, 3% of CO2 is added to the counting gas. The gamma ray penetrates a 0.01 inch of beryllium window before it enters to the counter. The beryllium window allows the entry of x rays into the detector. For this reason, materials with low atomic number Z are used and the window is thin. Also, the window must support the gas of the detector and it must resist pressure differences. For X rays up to several Angstroms, these conditions are met by thin foils of Al or Be. For lower photon energies, thin organic films are used, because they are fragile and semi-permeable. The cylindrical counter is made of iron. The operation voltage of the counters is: max. 2,0 kV, recommended 1,6 kV for Argon; max. 2,4 kV and recommended 2,0 kV for Xenon. Proportional counters have found several applications in the detection and energy measurement of low energy gamma rays and x-rays in the region up to 100 keV [5], [8], [9], [45].

1.9. Measurement of the energy of an incoming photon

The gamma photon is absorbed by the electron of an atom of the gas detector by the photoelectric effect. The photoelectron has the next kinetic energy: $K=E_\gamma-E_B$ where E_B is the bound energy of the electron ($E_B=U_o$). The bound energy is called work function. It is equal to the minimum energy $E_\gamma=hf_{min}$ necessary to eject an electron with kinetic energy almost with zero value: $E_\gamma=E_B$ $hf_{min}=E_B$ $f_{min}=E_B/h$ E_B=work function $K\approx0$ for f_{min}.

Afterward, the photoelectron loses its kinetic energy by causing further ionization (Bethe-Bloch). The number of ion-electron pairs produced is proportional to the energy of the electron. There is an average energy needed to ionize one atom and it depends on the gas detector used. The atom ionized by the photon has a vacancy in an inner shell and thus it emits a characteristic (fluorescence) X-ray or an Auger electron. But if an auger electron is emitted, then it also causes further ionizations. For other hand, the X-ray photon can make another photo effect or it can escape. If all energy is spent, then the number of electron-ion pairs is given by: $n=E_\gamma/\varepsilon$ where ε is the average energy needed to ionize one atom in the gas and E_γ is the energy of the X-ray photon [5], [8], [9], [45].

We have the recombination process when the voltage is not applied. If the voltage is increased, we have migration of the electrons to the anode wire (small radius). If the voltage is more increased, the electrons win more energy and it causes further ionizations. Therefore, there is an avalanche of electrons, and they soon arrives to the anode wire [5], [8], [9], [45]. The number of electrons arriving at the anode is: $N=n*\alpha$ where α is the amplification factor of the inner gas. Thus, N and the pulse height are proportional to E_γ.

1.10. The experiment

In the experiment, it is used as detector a proportional gas detector (proportional counter) and so the x-ray fluorescence spectra of the different materials are measured. It is used Americium (10 m Ci) as the radiative source and this source emits γ radiation. Americium (241) decays in Neptune (237). The main α energies emitted are 5.442 MeV (12,5%) and 5.484 MeV (85.2%) and the principal γ energy emitted is 59.5 keV (35,3%). It also emits some γ rays with energies from 11,9 to 22,2 keV. Therefore, we see that the most of the photons have energy of 60 keV [45].

This radiation is absorbed by a metallic sample, and it generates the characteristic fluorescence radiation of the target. The fluorescence radiation from our target leaves the apparatus through the hole, and then, we have Compton scattered photons which come through this hole. We put the apparatus in front of the window of our

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detector at approximately 5 cm. The fluorescence radiation of the target is emitted through a collimator and it is measured by the proportional counter. Afterwards, the signal of the detector is enhanced by a preamplifier and a main amplifier. The signal of the amplifier is sent to the multi-channel analyser (pulse height analyser), and thus, finally we can see the pulse spectrum in the PC computer. We measure from 10 to 30 minutes in order to get one spectrum [45].

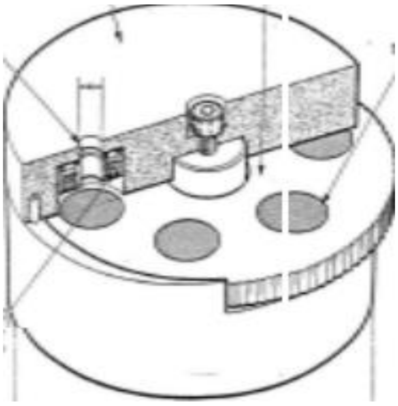

Fig.15. X-rays Source [45]

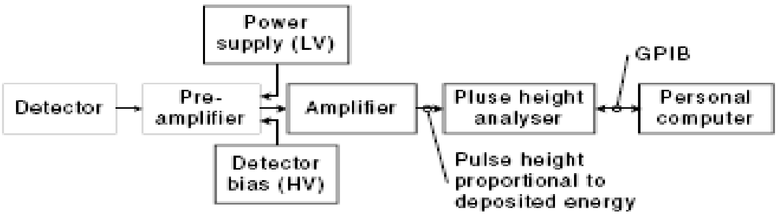

Fig.16. Experimental set-up [45]

1.11. Analysis of the experimental data

Determination of the energy calibration (k_α lines)

The gamma energies are a linear function of the channel numbers. The linear equation is determined based on the k_α peaks [6], [7], [45]. We have the next tables:

Table 1. Channel and K_α energy with the statistical error of the mean value for the channel number for Xe detector

Xe detector	Y (channel Xe)	X(Energy KeV Kα)	FWHM	σ	$\sigma/\sqrt{5}$
Rb	228	13,37	40	17,02	7,61
Mo	284	17,44	46	19,57	8,75
Ag	354	22,1	51	21,7	9,7
Ba	496	32,06	71	30,21	13,51
Tb	665	44,23	86	36,6	16,37

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Table 2. Channel and K_α energy with the statistical error of the mean value for the channel number for Ar detector

Ar detector	Y(channel Xe)	X (Energy KeV Kα)	FWHM	σ	$\sigma/\sqrt{3}$
Cu	144	8,04	30	12,77	7,37
Ag	339	22,1	40	17,02	9,83
Tb	650	44,23	152	65,11	37,59

If we want to know the parameters fitting the straight line $Y=a_1+a_2X$ and the uncertainties of the calibration function, which are described by the covariance ellipse (see item 1.17), then we apply the equations (A.2.12) and (A.2.14) (see item 1.17). Thus, Y is a matrix of the channel number of dimensions n x 1 and it contains the y values, a is the matrix that contains the unknown parameters: intercept a_1 and slope a_2 of dimensions 2 x 1, C of the dimensions n x 2 where all the elements of the first column are 1 and the second column contains the X values, and V is the diagonal covariance matrix for y_i, of dimensions n x n, where the elements of the diagonal are given by the formula: $\varepsilon^2=\sigma_i^2/n$ (where σ=FWHM/2.35) according to statistical laws (error propagation). Therefore, V^{-1} is the inverse of the covariance matrix (weight matrix) with elements in the diagonal given by the formula n/σ_i^2. In our case, it is n=5 for Xe and n=3 for Ar [6], [7], [45]. Therefore, if we resolve the matrix, we can find the parameters of the straight line, the uncertainties of the calibration function and the correlation coefficient related with the matrix (A.1.3) (see item 1.17) [6], [7], [45]. Thus, we have:

Table 3. Parameters, uncertainties, correlation coefficient and linear equation for Xe and Ar detectors

Detector	a1 channel	a2 channel/keV	σa1 channel	σa2 channel/keV	ρ	Linear Equation (Y:Ch)
Xe	37,76	14,27	11,44	0,51	-0,92	Y=37,76+14,27 X (E KeV)
Ar	31,75	14,02	11,54	0,71	-0,87	Y=31,75+14,02 X (E KeV)

We can plot our linear equation:

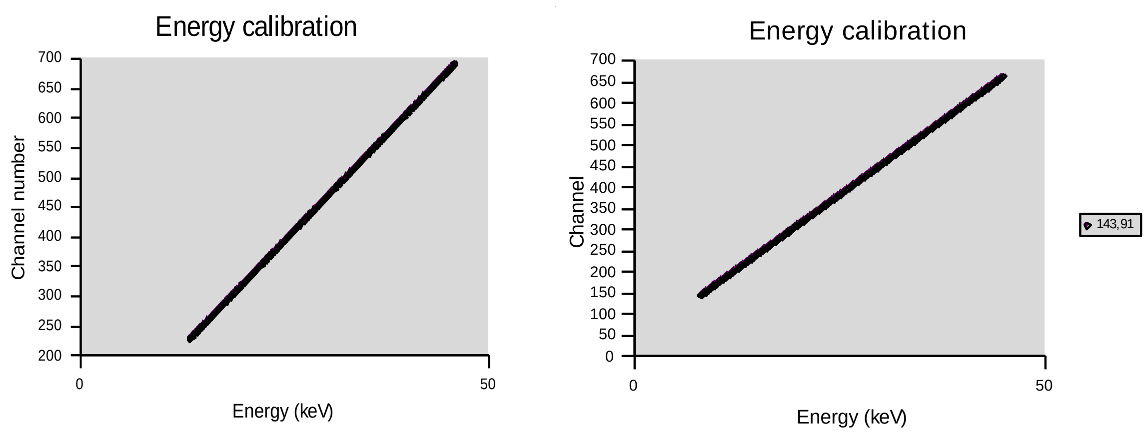

Fig.17. Energy calibration plot for Xe detector and for Ar detector

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In order to show the accuracy of the results for the values a_1 and a_2, we consider the covariance matrix for a (A.1.3) and the equation (A.1.4) (see item 1.17). It gives us a covariance ellipse in a plane spanned by the variables a_1 and a_2. We apply the equation of the ellipse (A.1.6) (see item 1.17).

It is the equation of an ellipse centered on the point (a_1, a_2). The principal axes of the ellipse make an angle with respect to the axes x_1 and x_2.

This angle and the half diameters ρ_1 and ρ_2 can be determined from equation (A.1.7) (see item 1.17), by using the properties of conic sections. So, we have the next table:

Table 4. Parameters of the ellipse for Xe and Ar detectors

Detector	α(°)	ρ1	ρ2
Xe	-2,35	11,43	0,2
Ar	-3,06	11,27	0,35

This ellipse is shown in the below figures for the fitted quantities a_1 (intercept) and a_2 (slope):

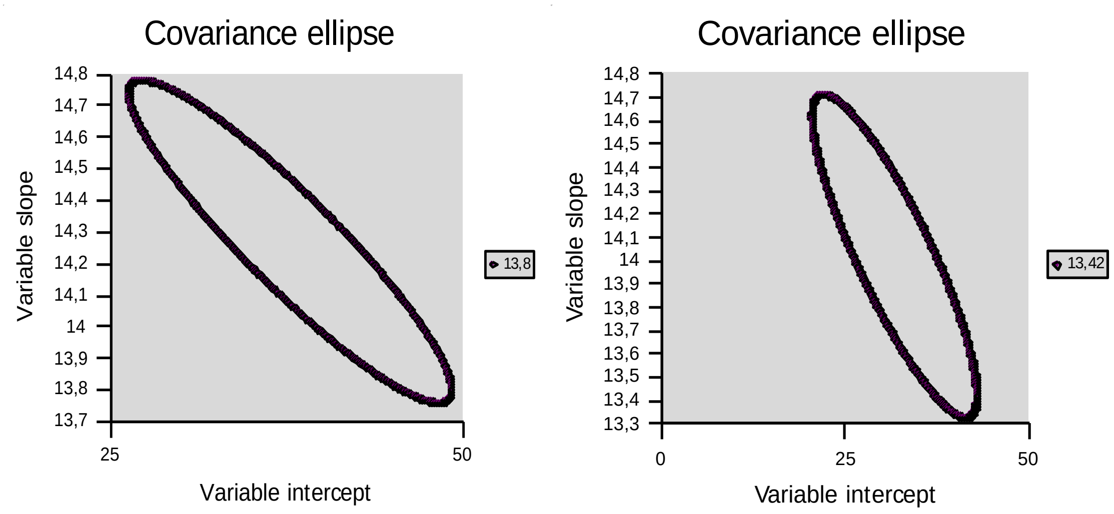

Fig.18. Covariance ellipse for Xe detector and for Ar detector

The points on the ellipse correspond to points of equal probability. Each of these points determines a line in the (x,y) plane. Thus, the points on the covariance ellipse correspond to a bundle of lines. The probability of observing a point (x_1, x_2) inside the ellipse is $1-(1/\sqrt{e})\approx 0.39$ where the integration region is given by the area within the covariance ellipse (see item 1.17).

Therefore, the line determined by the values of the unknown parameters lies in this bundle with this probability. The ellipse with these properties is called the covariance ellipse of the bivariate normal distribution [45].

1.12. Interpretation of the lines seen in the energy spectra

We consider for the experiment the spectrum for Tb, Ag and Cu in Argon detector:

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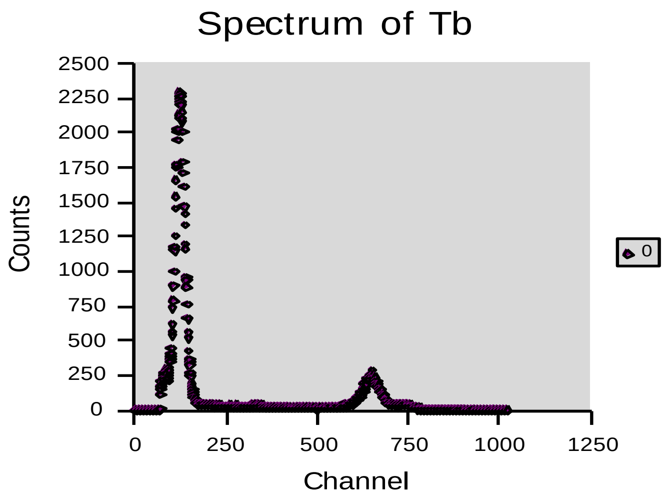

Fig.19. Spectrum of Tb

We can see our K_α line around the channel 650, the K_β line is almost distinguishable around the channel 741. Nevertheless, we have other line around the channel 124, and it comes from the Iron counter.

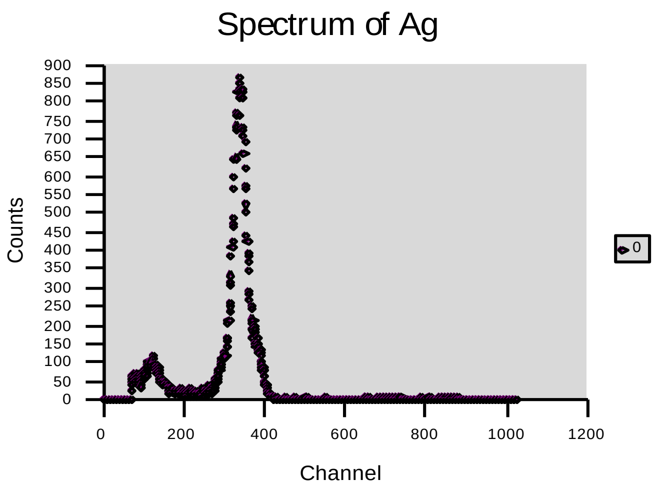

Fig.20. Spectrum of Ag

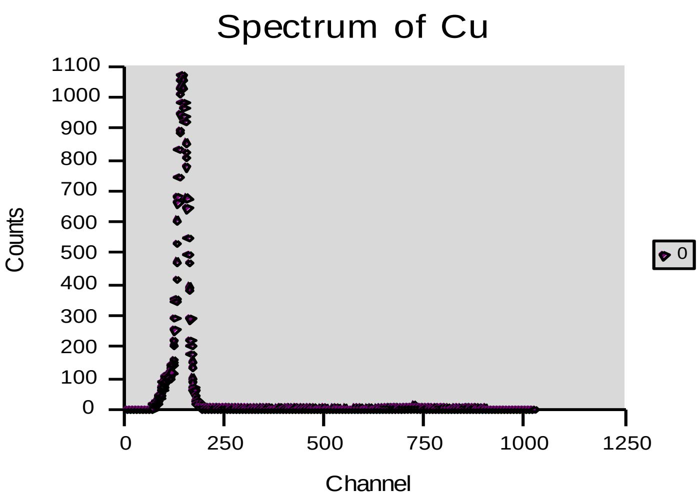

Fig.21. Spectrum of Cu

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We can see the K_α line around the channel 339, the K_β line is almost distinguishable around the channel 383. It is superposed to the K_α line, but it is possible to see if we separated from the K_α line by unfolding. We have other lines around of the channel 81 and 124 which correspond to lines from our Iron counter.

We can see in this spectrum the K_α line around the channel 144. The K_β line is not distinguishable.

1.13. Measurement of K_β energies

We have the next tables for Xe and Ar:

Table 5. K_β energies for Xe detector

Target	Kβ KeV	Channel
Tb	50,65	784
Ba	36,55	560
Ag	24,99	400
Cu	8,91	143

For Ar detector we have:

Table 6. K_β energies for Ar detector

Target	Kβ KeV	channel
Tb	50,65	741
Ag	24,99	383
Cu	8,91	Not distinguishable

1.14. Moseley's law

In order to apply the Moseley's law, we need to find the linear relation: $Y=a_1+a_2X$, where Y is E (Energy in keV for K_α lines) and X is $(Z-1)^2$. We have the next table for this case and for Xe detector:

Table 7. Table for Moseley's plot with the uncertainties of the experimental energy

Element	$(Z-1)^2$	Channel	Y (Energy alpha KeV)	σ	$\sigma_C=\sigma/\sqrt{5}$	σexp
Rb 37	1296	228	13,33	17,02	7,61	1,07
Mo 42	1681	284	17,26	19,57	8,75	1,118
Ag 47	2116	354	22,16	21,7	9,7	1,32
Ba 56	3025	496	32,11	30,21	13,51	1,69
Tb 65	4096	665	44,03	36,6	16,37	2,11

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We apply the error propagation method for:

X(Energy KeV)=[Y(channel)-a_1]/a_2 (item 1.11)

$$\sigma_{Exp} = (\sqrt{\sigma_{a1}{}^2 + (E_{exp}{}^2 * \sigma_{a2}{}^2) + \sigma_c{}^2})/a_2$$

where σ_c and E_{exp} (column Y) are given in the table 7 and the σ_{a1}, σ_{a2} and a_2 are showed in the table 3.

We want to determine the parameters of the intercept and the slope, and we can apply the same procedure of the item 11.1. In our case, V is the diagonal covariance matrix for y_i of dimensions 5x5 and the elements of the diagonal are $\sigma_{Eexp}{}^2$. Thus, V^{-1} is the inverse of the covariance matrix (weight matrix) and the elements in the diagonal are given by $1/\sigma_{Eexp}{}^2$.

Therefore, we have the next table by resolving the matrix:

Table 8. Moseley's plot: parameters, uncertainties, correlation coefficient, linear equation for Xe (X is $(Z-1)^2$)

Detector	a1 (KeV)	a2 (KeV)	σa1 (KeV)	σa2 (KeV)	ρ	Linear Equation
Xe	-1,97	0,01	1,6	$7,4*10^{-4}$	.0,93	Y(KeV)=-1,97+0,01X

We can plot our linear equation:

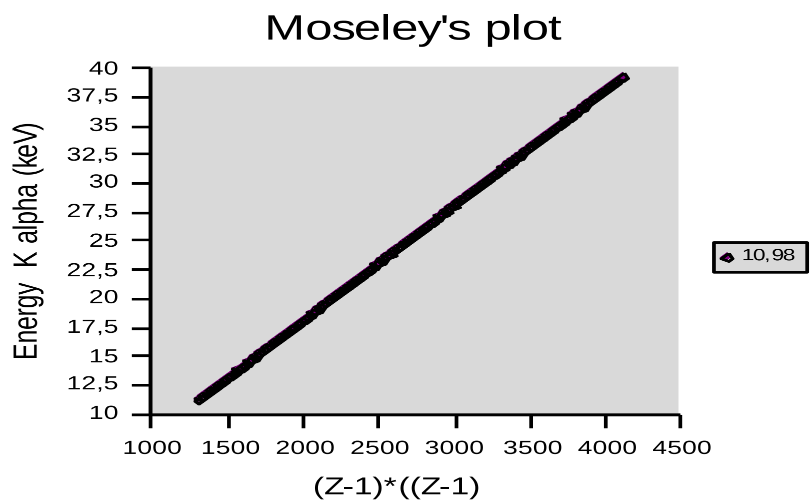

Fig.22. Moseley's plot for Xe detector

And for the covariance ellipse, we have the next parameters:

Table 9. Parameters of the ellipse for Xe detector and for the linear equation of the Moseley's law

Detector	α (º)	ρ1	ρ2
Xe	-0,025	1,63	$2,7*10^{-4}$

The plot of the covariance ellipse is given in the next figure:

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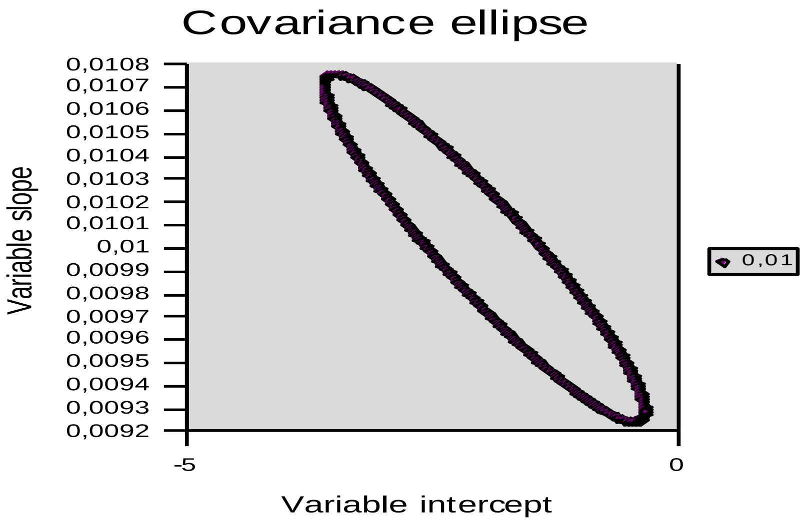

Fig.23. Covariance ellipse for Moseley's plot for Xe detector

The slope in the linear equation is 0,01*1000=$\frac{3}{4}$*R_y (energy) in accordance with the equation $E = h\nu_{K\alpha} = \frac{3}{4}13.6(Z-1)^2 eV$ and by comparing with the equation Y(E: keV)=-1.97+0.01(keV)X where X=$(Z-1)^2$.

Therefore, we have: R_y(energy)=13.33 eV which is a value very close to 13.6 keV

The error is obtained by the error propagation method:

$\sigma_{Ry}=\frac{4}{3}*\sigma_{a2}$ where $\sigma_{a2}=7.4*10^{-4}$ keV

σ_{Ry}=0.99 (eV)

The reference value of the Rydberg energy is 13,61 ev. It is obtained the relative error of (13.61-13.33)/13.61=2,06 %.

1.15. Comparison of the measured spectra of the two counters

We see that both detectors are quite transparent to their own characteristic K ray photons (K lines). For both detectors, we cannot compare line intensities directly. The fluorescence yield is higher than 75% and the escape peaks is higher than the original peak.

For photon energies above 20 keV, less than 10 % of the incoming radiation is absorbed in the gas and the photons which go through the gas are absorbed in the enclosure. Therefore, a part of the incoming quanta do not interact with the counting gas but inside the window or with the iron cylinder.

The photon absorption increases if we replace the argon detector by the xenon detector (due the better efficiency of xenon detector), but also it changes the range of the photoelectron. For other hand, we have many scape peaks with the Xe detector and it makes difficult the interpretation.

The x-ray rearrangement is of much lower energy with the argon detector, and thus, it is more easily retained in the gas but only at the expense of a much lower counting efficiency for the original photon.

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1.16. Conclusions

The statistical errors of the energy calibration (item 1.11) are possible to decrease if we have more elements in order to do the analysis. We can see that the correlation coefficient for Xe detector is higher than for Ar detector. It is due that we have less data for Ar detector. The correlation coefficient takes the value of +1 or -1 in the case of a perfect linear dependence (- sign is for the inverse relation). Therefore, we have very good dependence between the variables considered for Xe detector, and for Ar detector we need more data in order to improve the correlation coefficient and thus, the relation between the variables.

Besides, it is possible to improve our value of the Rydberg constant if we have more elements to consider for the linear fit of the Moseley's law. Nevertheless, our error in the slope is very small and our correlation coefficient is very near to 1. If we compare the experimental value (13.33 eV) with the reference value for the Rydberg energy (13.6 eV), we have the relative error of 2, 06% which is an acceptable value of error.

1.17. Covariance ellipse and the method of least squares in connection with the maximum likelihood

Covariance ellipse

The Gaussian distribution has the following properties [6], [7], [45]:

Probability density: $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-a)^2}{2\sigma^2}}$

Mean: E(x)=a

Variance: $\sigma^2(X)=\sigma^2$ (A.1.1)

The joint probability density of n random variables $\vec{X} = (X_1, X_2, \dots, X_n)$ is normal with the means $\vec{a} = (a_1, a_2, \dots, a_n)$ and the covariance matrix $C = B^{-1}$, if it has the form:

$$\varphi(\vec{x}) = ke^{-\frac{(\vec{x}-\vec{a})^T B(\vec{x}-\vec{a})}{2}} \qquad k = (2\pi)^{-n/2}(\det B)^{-1/2} \qquad \text{(A.1.2)}$$

Only if the covariance matrix is diagonal, then $\varphi(\vec{x})$ can be written as a product of n normal distributions with means $a_1, a_2, \dots,$ and variances $\sigma_1^2, \sigma_2^2, \dots.$

If $\vec{a} = (a_1, a_2)$ is a constant vector, then:

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} \sigma_1{}^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2{}^2 \end{pmatrix} \qquad B=C^{-1} \qquad \text{(A.1.3)}$$

If they are positive definite symmetric matrixes, then

$$\Phi(\vec{x}) = ke^{-\frac{(\vec{x}-\vec{a})^T B(\vec{x}-\vec{a})}{2}} \qquad \text{(A.1.4)}$$

$$k = \frac{1}{2\pi\sqrt{\det B}} \qquad \text{(A.1.5)}$$

Where, $\Phi(\vec{x})$ is the joint probability density of a normal distribution of the variables $\vec{X} = (X_1, X_2)$.

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The expectation values of the variables are a_1, a_2. Their covariance matrix is C. Lines of constant probability density in the x_1, x_2 plane correspond to constant values of the exponent. For a constant exponent, it is obtained the condition:

$$\frac{(x_1-a_1)^2}{\sigma_1{}^2} - 2\rho\frac{(x_1-a_1)(x_2-a_2)}{\sigma_1\sigma_2} + \frac{(x_2-a_2)^2}{\sigma_2{}^2} = constant. \qquad (A.1.6)$$

It is just the equation of an ellipse. For $(\vec{x}-\vec{a})^T B(\vec{x}-\vec{a}) = 1$, the right-hand side of the equation becomes constant=1-ρ^2 and the ellipse is called the covariance ellipse or error ellipse of the bivariate normal distribution. The error ellipse is centred at the point $\vec{a} = (a_1, a_2)$. For a correlation coefficient ρ=0 the principal axes of the error ellipse are parallel to the coordinate x_1, x_2 axes, and the principal semi-diameters of the ellipse ρ_1, ρ_2 are equals to σ_1, σ_2.

For $\rho\neq 0$, it is possible to find the principal axes and their orientations with respect to the coordinate axes from the formulas:

$$\rho_1{}^2 = \frac{\sigma_1{}^2\sigma_2{}^2(1-\rho^2)}{\sigma_2{}^2cos^2a - 2\rho\sigma_1\sigma_2 \sin a\cos a + \sigma_1{}^2 sin^2 a} \quad \& \quad \rho_2{}^2 = \frac{\sigma_1{}^2\sigma_2{}^2(1-\rho^2)}{\sigma_2{}^2 sin^2 a + 2\rho\sigma_1\sigma_2 \sin a\cos a + \sigma_1{}^2 cos^2 a}$$

$$\tan 2a = \frac{2\rho\sigma_1\sigma_2}{\sigma_1{}^2-\sigma_2{}^2} \qquad (A.1.7)$$

Where a is the angle between the x_1 axis and the semi-diameter of length ρ_1. Note that a is determined up to multiples of $\pi/2$.

The marginal distributions of the bivariate normal distribution are normal distributions of one variable:

$$g_i(x_i) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x_i-a_i)^2}{2\sigma_i{}^2}} \quad i=1,2 \qquad (A.1.8)$$

It is only for uncorrelated variables. For ρ=0, the bivariate normal distribution is the product of two univariate Gaussians:

$$\Phi(x_1,x_2)=g_1(x_1)g_2(x_2) \qquad (A.1.9)$$

The correlation coefficient between two random variables X_i and X_j is the covariance of the variables divided by the square root of the product of the variances (Variance: $\sigma^2(X)=\sigma^2$):

$$\rho_{ij} = \frac{C_{ij}}{\sqrt{(C_{ii}C_{jj})}} = \frac{cov(X_i,X_j)}{\sigma(X_i)\sigma(X_j)} \qquad (A.1.10) \qquad (C_{11}=\sigma_1{}^2 \quad C_{22}=\sigma_2{}^2)$$

It has the range $-1\leq\rho_{ij}\leq 1$ and vanishes for independent variables. If ρ_{ij} =1, X_i and X_j have a perfect linear dependence and the covariance matrix is singular. The correlation coefficient can be regarded as a measure of the relation between the statistical distributions of the two random variables considered.

The C_{ij} constitute the elements of the covariance matrix and the covariance matrix is always symmetric.

If all n variables are independent, its determinant is zero. If the determinant is zero, it means that a perfect linear relation exists between the variables [6], [7], [45]. The covariance ellipse always lies inside a rectangle determined by the point (a_1,a_2), and the standard deviations σ_1 and σ_2. It touches the rectangle at four points. For the extreme case ρ=(+/-) 1the ellipse becomes one of the two diagonals of this rectangle.

From the equation (A.1.6), it is clear that other lines of constant probability (for $(\vec{x}-\vec{a})^T B(\vec{x}-\vec{a})$ different to 1) are also ellipse, concentric and similar to the covariance ellipse and situated inside (or outside) of it for larger (or smaller) probability.

Therefore, the bivariance normal distribution corresponds to a surface in the three dimensional space (X_1, X_2,) whose horizontal sections are concentric ellipses. For the largest probability, this ellipse collapses to the point (a_1,a_2). The vertical sections through the center have form of a Gaussian distribution whose width is directly proportional to the diameter of the covariance ellipse [6], [7], [45].

The method of least squares in connection with the maximum likelihood

If measurements *y* have been performed, and p(y/x)is the normalized ($\int pdy = 1$) probability density of y as function of parameters x, then the parameters *x* can be estimated by maximizing the joint probability density for the m measurements y_j (assumed to be independent)

$$L\left(\frac{y}{x}\right) = \prod_{j=1}^{m} p\left(\frac{y_i}{x}\right) \qquad \text{(A.2.1)}$$

L(y/x)=L(y_1,….,y_m/x) is called the likelihood function.

L is a measure of the probability of observing the particular sample y given x. If p(y/x) is a normal distribution and if its variance is independent of the parameters x, then the maximum-likelihood method is identical to the least squares method.

The general problem is often solved numerically by the minimization of -ln (L). The data sample consists of N pairs of points (X_i, Y_i), where the X_i are known exactly, while the Y_i have been measured with resolution σ_i with independent errors [6], [7], [45].

If we assume that the errors are gaussian and Y is given by a theoretical function $Y = f\left(X;\ a_1, \ldots, a_p\right)$, then the individual probability for the point i is given by:

$$P(y_i;a) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y_i - f(x_i,a))^2}{2\sigma_i^2}} \qquad \text{(A.2.2)}$$

$$L\left(\frac{y}{x}\right) = \prod_{j=1}^{m} p\left(\frac{y_i}{x}\right) \qquad L\left(\frac{y}{x}\right) = \prod_{j=1}^{m} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y_i - f(x_i,a))^2}{2\sigma_i^2}}$$

and the logarithm of the likelihood is:

$$\ln L = -\frac{1}{2}\sum_i \frac{(y_i - f(x_i,a))^2}{\sigma_i^2} - \sum_i \ln(\sqrt{2\pi}\sigma_i) \qquad \text{(A.2.3)}$$

The Maximum likelihood method consists of minimizing the quantity ln L, which corresponds to:

$$\sum_i \left(\frac{y_i - f(x_i,a)}{\sigma_i}\right)^2 \equiv x^2 \qquad \text{(A.2.4)}$$

with respect to each one of the parameters a_k:

$$\forall k, \sum_i \frac{\partial}{\partial a_k}\left[\frac{y_i - f(x_i, a)}{\sigma_i}\right]^2 = 0 \qquad \text{(A.2.5)}$$

The use of the x^2notation in the above definition, is because x^2 is distributed according to the x^2 distribution.

If the observations are correlated with errors and covariance terms given in the covariance matrix V=V(y), then the Least Square Principle for estimating the parameters is formulated as:

$$x^2 = \sum_{i=1}^{N}\sum_{j=1}^{N}(y_i - f_i)\, {V_{ij}}^{-1}(y_j - f_j) = minimum \quad \text{(A.2.6)}$$

If many parameters are to be estimated in a linear fit, then the matrix notation produces elegant results. It is denominated:

$\underline{a} = a_1, \ldots, a_p$: the vector of the p parameters.

$\underline{y} = y_1, \ldots, y_N$: the vector of the N measured quantities and similarly for x.

The fit will be called linear if the theoretical function can be expressed as a linear combination of any function of x:

$$f(x, \underline{a}) = \sum_{r=1}^{p} c_r(x) a_r \qquad \text{(A.2.7)}$$

In the general case, the x^2 can be expressed as a sum of exponents of the multinormal functions:

$$x^2 = \sum_i \sum_j (y_i - f(x_i; \underline{a})){V_{ij}}^{-1}(y_j - f(x_j; \underline{a})) \quad \text{(A.2.8)}$$

which can be expressed in matrix notation:

$$x^2 = (\underline{y} - \underline{f})^T V^{-1}(\underline{y} - \underline{f}) \qquad \text{(A.2.9)}$$

Assuming that the y´$_i$ are independent, each of them with an uncertainty σ_i, it is possible to write:

$$x^2 = \sum_i^N \left[\frac{y_i - \sum_{r=1}^{p} a_r c_r(x_i)}{\sigma_i}\right]^2 \qquad \text{(A.2.10)}$$

By doing the differentiation of this expression with respect to a_r , it gives the normal equations:

$$\sum_i c_r(x_i) \frac{y_i - \sum_s \widehat{a_s}\, C_s(x_i)}{{\sigma_i}^2} = 0 \qquad \text{(A.2.11)}$$

It can be expressed in matrix notation, introducing the matrix:

$$C_{ir} = c_r(x_i)$$

$$\underline{f} = C\underline{a}$$

$$C^T V^{-1} C \underline{\hat{a}} = C^T\, V^{-1} \underline{y}$$

$$\underline{\hat{a}} = (C^T V^{-1} C)^{-1} C^T V^{-1} \underline{y} \qquad \text{(A.2.12)}$$

Where, a is the vector of the p parameters.

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The dimensions of the various matrixes are:

$\underline{y}$: Nx1

$\underline{a}$: px1

C: Nxp

V: NxN

$C^T V^{-1}$: pxN

$C^T V^{-1} \underline{y}$: px1

$C^T V^{-1} C$: pxp

So, the Linear Least Square Method provides an unique solution, which is linear with respect to the observables y_i.

Secondly, this estimator is unbiased because it is linear:

$$
\begin{aligned}
<\underline{\hat{a}}> &= <(C^T V^{-1} C)^{-1} C^T V^{-1} \underline{y} > \\
&=(C^T V^{-1} C)^{-1} C^T V^{-1} < \underline{y} > \\
&=(C^T V^{-1} C)^{-1} C^T V^{-1} C \underline{a} \\
&= \underline{a}
\end{aligned}
$$

We are going to calculate the covariance matrix for a. If we establish z=Mx, then we can apply the error propagation method.

The covariance matrix for z is given by:

$$V(z)=MV(x)M^T \qquad (A.2.13)$$

Then, in our case we have:

a=M y where: $M= (C^T V^{-1} C)^{-1} C^T V^{-1}$

$$V(\underline{\hat{a}})= (C^T V^{-1} C)^{-1} C^T V^{-1} V V^{-1} C (C^T V^{-1} C)^{-1} =(C^T V^{-1} C)^{-1} \qquad (A.2.14)$$

2. Compton Effect

At the macroscopic world, all the properties exhibited by the light are wave properties: refraction, reflection, diffraction, and interference. Besides, Maxwell's equations predict the existence of electromagnetic waves produced by oscillating charges and the wave properties seem to be a perfect way to describe light and other forms of electromagnetic radiation, at least at the macroscopic level [10], [14], [15], [16], [46].

However, the electromagnetic radiation exhibits a different set of properties at the microscopic level. At this level, the photoelectric effect showed that light in the nature is granular rather than smooth and that it carries its energy in discrete bundles (packages) called photons. Therefore, the interaction between the electromagnetic radiation and the matter must be described in terms of the interaction between the individual photons and the individual particles (electrons) at the microscopic scale [10], [14], [15], [16], [46].

Nevertheless, the Compton effect provides more information about the photon properties than the photoelectric effect. The photon carries a specific amount of momentum and energy. Therefore, certain types of scattering of electromagnetic radiation with the matter can be described in terms of collisions between the individual photons and the individual particles (electrons). The standard rules of relativistic kinematics apply to these collisions, and it can be used to determine the properties of the scattered radiation. The Compton effect shows that in these collisions, photons behave like particles with a rest mass of zero. Thus, the electromagnetic radiation is essentially composed of streams of particles called photons at the microscopic scale [10], [14], [15], [16], [46].

When a monochromatic X-ray beam strikes a target, it may be absorbed (photoelectric effect) or scattered from the atomic nuclei or the electrons.

Scattering by the atomic electrons will occur when an X-ray photon collides with one of the electrons of the absorbing element, and the photon is deflected from its original direction with or without loss of energy (Compton or Rayleigh scattering respectively). At low energy of the incident X-ray photon, there is only scattering of the incident photon and no ionization without loss of energy of the incident X-ray photon [10], [14], [15], [16], [46].

Firstly, we describe the coherent or Rayleigh scattering. It occurs when an electromagnetic wave moves near an electron and excites it into oscillations with the same frequency and with the same phase as the incident photon. This collision is elastic because there is not energy lost in this process. Since there is not energy change involved, the coherent scattered radiation will retain the same wavelength as the incident beam [10], [14], [15], [16], [46]. The probability of this effect is proportional to $Z^{2.6}/E^2$.

Also, it can happen that the scattered photon gives up a small part of its energy to the electron during the collision, especially when the electron with who the photon collides is loosely bound. It occurs at energies much greater than the binding energies of the electrons and the photons are scattered from the electrons as if the electrons are free (loosely bound) and at rest.

In this instance, the scatter is said to be incoherent, and no elastic called Compton scattering and the wavelength of the incoherent scattered photon λ is greater than λ_o. Thus, $\lambda>\lambda_o$ is because the wavelength is given by the formula $\lambda=hc/E$ and the energy of the scattered photon has decreased $E<E_o$ and also the frequency has decreased $f<f_o$ because $E=hf$ where h is the Planck constant. It is the Compton effect: the most probable process for photons in the intermediate energy range and the probability decreases rapidly with the increasing energy. The probability of this effect is proportional to Z, the atomic number [10], [14], [15], [16], [46].

Therefore, an incident γ ray scatters from an outer shell electron of the absorber material at an angle Φ. Part of the γ ray energy is imparted to the electron as kinetic energy and the photon is scattered at an angle Φ as it is showed in the next figure. Typically, the photon will have sufficient energy to produce several recoil electrons and all possible energy losses will occur. The net result is the production of ions cascade as fast electrons react with other atoms [10], [14], [15], [16], [46].

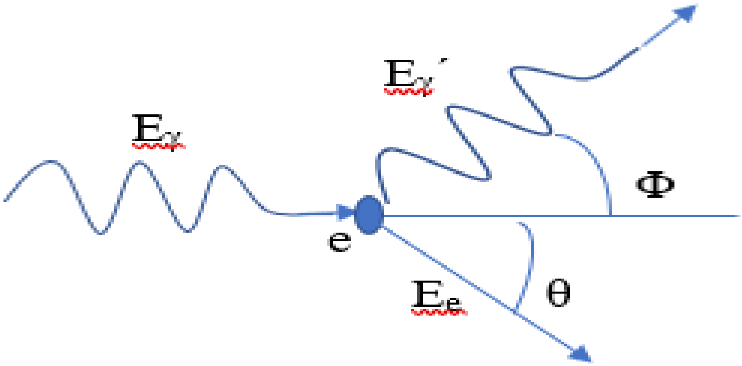

Fig.24. The Compton interaction

The conservation of the energy and the momentum leads us to the following expression for wavelength and the energy of the scattered photon:

$\Delta\lambda = \lambda' - \lambda = \frac{h}{m_o c}(1-\cos\Phi)$ **Compton wavelength formula**

$E'_\gamma = \frac{E_\gamma}{1+\frac{E_\gamma}{m_o c^2}(1-\cos\Phi)}$ **Energy of the scattered photon**

Where, E_γ is the incident photon energy, E_γ' is the energy of the scattered photon, Φ is the scattering angle of the photon and m_oc^2 is the rest mass energy of the electron. The kinetic energy and the total energy of the electron after the collision are given by:

$K = E_\gamma - E'_\gamma = hf - hf' = \frac{{E_\gamma}^2(1-\cos\Phi)}{m_o c^2 + E_\gamma(1-\cos\Phi)}$ **(kinetic energy of the electron)**

$E_e' = m_oc^2 + K = m_oc^2 + \frac{{E_\gamma}^2(1-\cos\Phi)}{m_o c^2 + E_\gamma(1-\cos\Phi)}$ **(total energy of the electron)**

It can be seen that since all scattering angles are possible, the kinetic energy of the electron ranges from 0 for $\Phi=0^o$ to $2E_\gamma^2/(m_oc^2+2E_\gamma)$ (maximum energy which can be transferred to the electron) for $\Phi=180^o$.

For other hand, the photon never loses the whole of its energy in any one collision. The scattered photon can then continue through the absorber material and interact again or scatter out of the absorber material completely.

If the full energy of the incident photon is not absorbed in the detector, then there is a continuous background in the energy spectrum, known as the Compton continuum. This continuum extends up to an energy corresponding to the maximum energy transfer, where there is a sharp cut-off point, known as the Compton edge ($\Phi=180^o$: backscattering of the photon). There is a probability that each event has approximately equal chance to produce a pulse with any height up to this maximum. Therefore, Compton events will provide a well distributed low-energy area in the spectrum [10], [14], [15], [16], [46].

Deriving Compton's Formula

The standard rules of relativistic kinematics apply to these collisions, and it can be used to determine the properties of the scattered radiation as it was mentioned before.

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Momentum Conservation

$p_\gamma = p_\gamma' \cos\Phi + p_e' \cos\theta$ $p_e=0$ because initially the electron is at rest.

$0 = p_\gamma' \operatorname{sen}\Phi - p_e' \operatorname{sen}\theta$

$p_\gamma - p_\gamma' \cos\Phi = p_e' \cos\theta$

$p_\gamma' \operatorname{sen}\Phi = p_e' \operatorname{sen}\theta$

By squaring both sides of the two equations and adding both equations, it is obtained:

$p_\gamma^2 - 2\, p_\gamma\, p_\gamma' \cos\Phi + p_\gamma'^2 = p_e'^2$

Energy Conservation

$m_o c^2 + E_\gamma = E_e' + E_\gamma'$ $m_o = m_e$: rest mass of the electron $p_e=0$

$E_e' = m_o c^2 + E_\gamma - E_\gamma'$ $K = E_\gamma - E_\gamma'$

$E_e' = m_o c^2 + K$ Total Energy for the electron

By using the Dirac equation and the total energy for the electron, it is obtained:

$E_e'^2 = p_e'^2 c^2 + m_o^2 c^4$ Dirac Equation

$E_e' = m_o c^2 + K$ Total Energy for the electron

$(m_o c^2 + K)^2 = p_e'^2 c^2 + m_o^2 c^4$

$m_o^2 c^4 + 2\, m_o c^2 K + K^2 = p_e'^2 c^2 + m_o^2 c^4$

$2\, m_o c^2 K + K^2 = p_e'^2 c^2$

$2\, m_o K + \frac{K^2}{c^2} = p_e'^2$

$p_\gamma^2 - 2\, p_\gamma\, p_\gamma' \cos\Phi + p_\gamma'^2 = p_e'^2$

$2\, m_o K + \frac{K^2}{c^2} = p_\gamma^2 - 2\, p_\gamma\, p_\gamma' \cos\Phi + p_\gamma'^2$

$p_\gamma = h/\lambda$ $p_\gamma' = h/\lambda'$ $K = E_\gamma - E_\gamma'$ $K = hf - hf' = (hc/\lambda) - (hc/\lambda')$ $f = c/\lambda$ $f' = c/\lambda'$

$$2\, m_o\left(\frac{hc}{\lambda} - \frac{hc}{\lambda'}\right) + \frac{\left(\frac{hc}{\lambda} - \frac{hc}{\lambda'}\right)^2}{c^2} = \left(\frac{h}{\lambda}\right)^2 - 2\left(\frac{h}{\lambda}\right)\left(\frac{h}{\lambda'}\right)\cos\Phi + \left(\frac{h}{\lambda'}\right)^2$$

$$2\, m_o hc\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right) + \left(\frac{h}{\lambda}\right)^2 - 2\left(\frac{h}{\lambda}\right)\left(\frac{h}{\lambda'}\right) + \left(\frac{h}{\lambda'}\right)^2 = \left(\frac{h}{\lambda}\right)^2 - 2\left(\frac{h}{\lambda}\right)\left(\frac{h}{\lambda'}\right)\cos\Phi + \left(\frac{h}{\lambda'}\right)^2$$

$$2\, m_o hc\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right) = 2\left(\frac{h}{\lambda}\right)\left(\frac{h}{\lambda'}\right)(1 - \cos\Phi)$$

$$m_o c\left(\frac{\lambda' - \lambda}{\lambda'\lambda}\right) = \frac{h}{\lambda'\lambda}(1 - \cos\Phi)$$

$\Delta\lambda = \lambda' - \lambda = \frac{h}{m_o c}(1 - \cos\Phi)$ **Compton wavelength formula**

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$m_o = m_e$: rest mass of the electron, λ: wavelength of incident radiation λ´: wavelength of scattered radiation, $\lambda_c = \frac{h}{m_o c} = 0{,}0243 \cdot 10^{-10}$ m is the Compton wavelength.

$\frac{hc}{E'} - \frac{hc}{E} = \frac{h}{m_o c}(1 - \cos\Phi)$ E=hf=hc/λ λ=hc/E

$E'_\gamma = \frac{E_\gamma}{1 + \frac{E_\gamma}{m_o c^2}(1 - \cos\Phi)}$ **Energy of the scattered photon**

Where, E_γ is the incident photon energy, $E_\gamma{}'$ is the energy of the scattered photon.

The kinetic energy of the electron after the collision is given by:

$$K = E_\gamma - E'_\gamma = E_\gamma - \frac{E_\gamma}{1 + \frac{E_\gamma}{m_o c^2}(1 - \cos\Phi)}$$

$K = E_\gamma - E'_\gamma = \frac{E_\gamma{}^2(1 - \cos\Phi)}{m_o c^2 + E_\gamma(1 - \cos\Phi)}$ **Kinetic energy of the electron**

where: $E_e{}' = m_o c^2 + K$ $E_{e'} = m_o c^2 + hf - hf'$ $K = hf - hf'$ $m_o = m_e$

$E_e{}' = m_o c^2 + \frac{E_\gamma{}^2(1 - \cos\Phi)}{m_o c^2 + E_\gamma(1 - \cos\Phi)}$ **total energy of the electron**

Deriving Compton's Formula by four momentum conservation

The Compton experiment offers a proof of the particle-like behavior of the light. It is possible to show by applying the conservation of energy and momentum to the collision between a photon and an electron. By applying the four-momentum conservation, it is obtained:

$q_\gamma + q_e = q_\gamma{}' + q_e{}'$

$q_\gamma - q_\gamma{}' = q_e{}' - q_e$

$q_\gamma{}^2 + q_\gamma{}'^2 - 2q_\gamma{}' q_\gamma = q_e{}^2 + q_e{}'^2 - 2q_e q_e{}'$

$q_\gamma = \begin{pmatrix} E_\gamma \\ \vec{p_\gamma} \end{pmatrix}$ $q_\gamma{}^2 = E_\gamma{}^2 - p_\gamma{}^2 = 0$ $q_\gamma{}'^2 = E_\gamma{}'^2 - p_\gamma{}'^2 = 0$ $c^2 = 1$

$q_e = \begin{pmatrix} E_e \\ \vec{p_e} \end{pmatrix}$ $q_e{}^2 = E_e{}^2 - p_e{}^2 = m_o{}^2$ $q_e{}'^2 = E_e{}'^2 - p_e{}'^2 = m_o{}^2$ $c^2 = 1$ (m_o: electron mass)

$-2(E_\gamma{}' E_\gamma - p_\gamma{}' p_\gamma \cos\Phi) = 2 m_o{}^2 - 2E_e E_e{}'$ $p_e = 0$ (the electron is initially at rest)

$E_e = m_o$ $E_e{}' = m_o + K = m_o + hf - hf'$ $K = hf - hf'$

$E_\gamma = hf$ $E_\gamma{}' = hf'$

$p_\gamma = E_\gamma / c$ $p_\gamma = E_\gamma$ $p_\gamma{}' = E_\gamma{}'/c$ $p_\gamma{}' = E_\gamma{}'$ (c=1)

$-2\,hf'\,hf(1 - \cos\Phi) = 2\,m_o{}^2 - 2m_o(m_o + hf - hf')$

$-hf'\,hf(1 - \cos\Phi) = -\,m_o(hf - hf')$

$\frac{h}{m_o c^2}(1-\cos\Phi)=\frac{f-f'}{ff'}$ (by reintroducing the factor c^2) $c=\lambda f$ $f=c/\lambda$

$\frac{h}{m_o c}(1-\cos\Phi)=\lambda'-\lambda$

$\Delta\lambda=\lambda'-\lambda=\frac{h}{m_o c}(1-\cos\Phi)$

Therefore, it is obtained: $\Delta\lambda=\lambda'-\lambda=\lambda_c(1-\cos\Phi)$ where λ' is the wavelength of the scattered x-rays, λ is the wavelength of the incident x-rays, $\lambda_c = h/m_o c$ λ_c=0,0243 A, constant known as the Compton wavelength for the electron.

If we replace $E = hf = hc/\lambda$ $\lambda=hc/E$ $\lambda'=hc/E'$, it is possible to get the expressions for the energy of the scattered photon and the kinetic energy of the electron:

$E'_\gamma=\frac{E_\gamma}{1+\frac{E_\gamma}{m_o c^2}(1-\cos\Phi)}$ **Energy of the scattered photon**

Where, E_γ is the incident photon energy, E_γ' is the energy of the scattered photon, Φ is the scattering angle of the photon and $m_o c^2$ is the rest mass energy of the electron. The kinetic energy of the electron after the collision is given by:

$K=E_\gamma-E'_\gamma=hf-hf'$

$K=E_\gamma-E'_\gamma=E_\gamma-\frac{E_\gamma}{1+\frac{E_\gamma}{m_o c^2}(1-\cos\Phi)}$ **(kinetic energy of the electron)**

$K=E_\gamma-E'_\gamma=\frac{{E_\gamma}^2(1-\cos\Phi)}{m_o c^2+E_\gamma(1-\cos\Phi)}$

The total energy of the electron after the collision is given by:

$E_{e'}=m_o c^2+K$

$E_e'=m_o c^2+\frac{{E_\gamma}^2(1-\cos\Phi)}{m_o c^2+E_\gamma(1-\cos\Phi)}$ **(total energy of the electron)**

It was assumed in the derivation of the Compton wavelength shift equation, that the electron involved was free (loosely bound), rather than being bound to an atom. If the electron had been bound to the atom, the situation would have been quite different, since also the atom would have been involved in the collision. The assumption is valid for electrons lightly bound with energies of few eV and for photon energies much bigger than it, in the order of keV or more [46].

Klein-Nishina formula

The Klein Nishina formula provides an accurate prediction of the angular distribution of x-rays and gamma rays which are incident upon a single electron. The Klein-Nishina formula describes the incoherent or Compton scattering. It provides the differential cross section with respect to the solid angle of scattering dΩ and accounts for factors such as the radiation pressure and the relativistic quantum mechanics. For an incident photon of energy

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E_γ=hf and by using the factor $\alpha=E_\gamma/(m_oc^2)$, it is obtained:

$$\frac{d\sigma_c}{d\Omega}=\frac{r_o^2}{2}\frac{1}{[1+\alpha(1-\cos\Phi]^2}[1+\cos^2\Phi+\frac{\alpha^2(1-\cos\Phi)^2}{1+\alpha(1-\cos\Phi)}]$$

Where, Φ is the scattering angle of the photon, r_o=2.82 fm is the classical electron radius, m_o is the mass of the electron.

The value $d\sigma_c/d\Omega$ is the scattering photon into the solid angle defined by dΩ=2πsinΦdΦ. The differential cross section depends at the energy of the incoming photon and the scattering angle. This formula was derived in 1929 by Oscar Klein and Yoshio Nishina. It was one of the first results obtained from the study of quantum electrodynamics. The consideration of the relativistic quantum mechanical effects allowed the development of an accurate equation for the radiation scattering from the electron target. Before this derivation, J.J. Thomson, British physics and discoverer of the electron, derived the electron cross section. However, scattering experiments showed significant deviations from the results predicted by the Thomson cross section. Further scattering experiments agreed perfectly with the prediction of the Klein Nishina formula [10], [14], [15], [16], [46].

2.1. Compton's experiment

Arthur Compton performed his experiment in 1923 and at this time, it had already been known that a material illuminated by x-rays gave off what were called secondary rays. The objective of the Compton experiment was to show that these secondary rays were primarily the result of scattering of the incident x-rays from electrons in the material [46]. The scattering of x-rays from free electrons was explainable in terms of the classical electromagnetic wave theory of radiation. The details were worked out by Thomson. Besides, J.A. Gray noticed that the scattered x-rays had the same polarization and roughly the same intensity as were predicted by Thomson's scattering theory, but that the scattered rays were absorbed more readily than the incident x-rays. Therefore, the energy of the scattered photon has decreased $E<E_o$. It was verified by Compton, and it occurred to him, that this increased absorbability could be explained if one assumed that the wavelength of the scattered x-ray was slightly higher than the original wavelength of the incident x-rays. The wavelength is given by the formula λ=hc/E and $E<E_o$, then $\lambda>\lambda_o$. The measurements done by Compton of the absorbability of the scattered x-rays over a wide range of incident wavelengths, indicated that the increase in the wavelength of the scattered x-rays was consistent in the order of 0.03 A. Then, Compton decided to check this wavelength increase directly, using an x-ray spectrometer. Besides, Compton checked if the wavelength shift was depended upon the angle Φ through which the x-rays were scattered [46]. Compton's experimental set-up is shown schematically at the next figure:

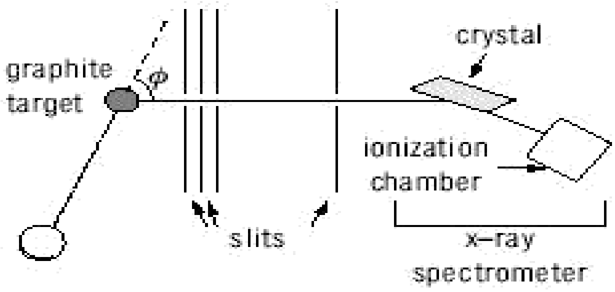

Fig.25. Compton´s experimental setup [46]

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X-rays of known wavelength were produced in an x-ray tube and allowed to strike a graphite target. A series of slits allowed entering to the spectrometer to only those scattered x-rays which left the target in a direction making an angle Φ with the direction of the beam of incident x-rays. This angle Φ was the angle through which these particular X-rays had been scattered and its value could be varied by moving the x-ray source [46].

The X-ray spectrometer consisted of a crystal from which the X-rays were reflected and directed to an ionization chamber where the X rays were detected. The wavelength of the scattered X-rays could be determined from the angle at which they were reflected from the crystal with maximum intensity [46].

The next figure shows the intensity of the scattered X-rays as a function of the wavelength.

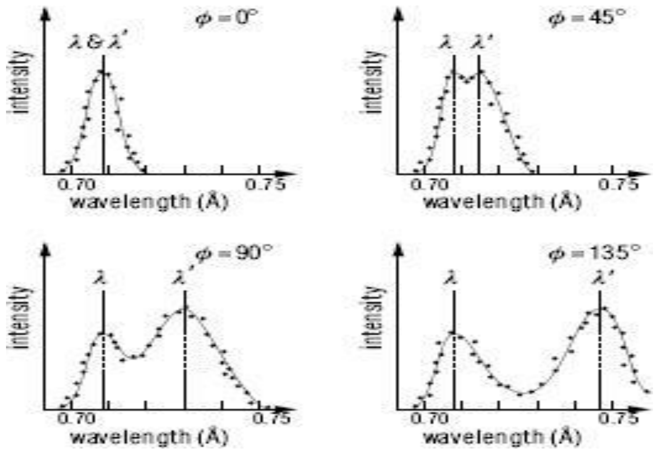

Fig.26. Intensity of the scattered x rays versus the wavelength [46]

The output of the spectrometer for several values of the scattering angle Φ is shown in the figure. The incident X-rays on the graphite target have a wavelength of λ = 0.707 A. When Φ = 0°, the detected X-rays in the spectrometer are essentially those which have undergone with no scattering. Thus, the spectrometer's output is a single peak centered on λ =0.707 A at this angle [46].

As the value of Φ is increased, the single peak splits up into two peaks, one at the original value of λ =0.707 A, and the other at the increased wavelength λ ' whose value depends upon the value of Φ. Compton showed that at least some of the scattered x-rays had their wavelengths increased in the scattering process.

For other hand, the amount by which the wavelength of a scattered x-ray increased was directly related to the angle Φ through which it had been scattered. The empirical relationship (obtained by mean of the experimental results) for the shifted peak position was given by: $\Delta\lambda=\lambda ' - \lambda = \lambda_c (1-\cos\Phi)$ where λ' is the wavelength of the scattered X-rays, λ is the wavelength of the incident X-rays, $\lambda_c = h/m_o c = 0{,}0243$ A, constant known as the Compton wavelength for the electron, m_o is the electron mass, h is the Planck constant, c is the light velocity and Φ is the angle through which X-rays are scattered [46].

2.2. Experimental Setup

The next figure shows the experimental set-up. It is used a NaI(Tl) scintillator crystal. It will stop some of the scattered photons from the Al pole. The photon source is Cs137 (with E_γ = 662keV) which is partially enclosed in a lead chamber which serves as a collimator [11], [12], [13], [17], [46].

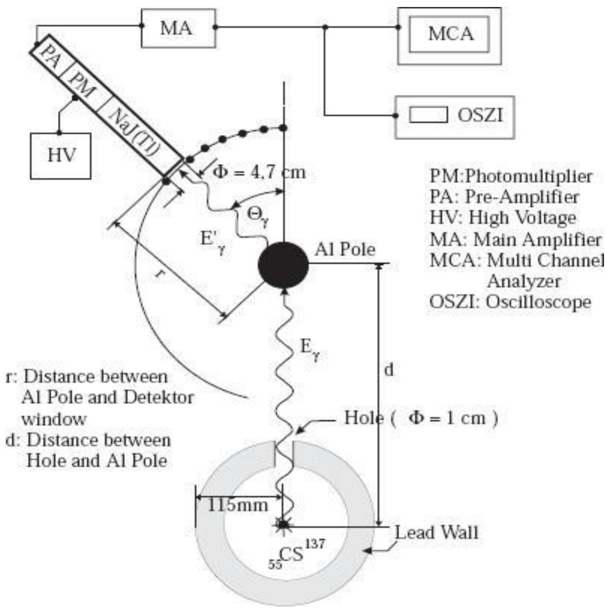

Fig.27. Experimental Setup [46]

The photomultiplier, preamplifier and the detector (NaI(Tl) scintillator crystal) are enclosed together in an aluminum shell. The detector is located on a plywood support, and so it can be moved to cover the angles from -90 to +90 degrees. Besides, it can be fixed in positions of intervals of 15 degree.

A narrow channel drilled through the center of the lead cylinder (where is positioned the radioactive source) allows to scape only photons which travel along it in order to scatter the target. The direction of the channel defined by an angle of 0^o is the direction of the un-scattered beam. The emerging photons (from the lead cylinder) impact at the Al Pole target, and only some small fraction of the incident photons will be scattered. Thus, some of the scattered photons will enter to the NaI(Tl) crystal which is attached to the photomultiplier tube (PMT) [11], [12], [13], [17], [46].

The reasons for choosing the Cs 137 source are:

1.- The effect is well observable at an energy E_γ = 662keV.

2.- The emission spectrum is mainly monoenergetic as it can be seen from the decay scheme in the next figure.

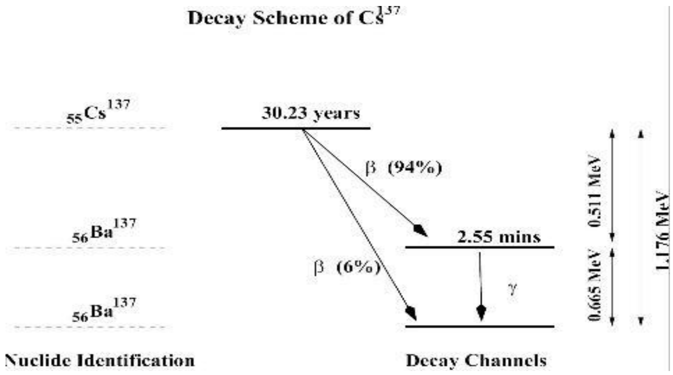

Fig.28. Decay scheme of Cs137 [46]

A Thallium doped Sodium iodine crystal NaI(Tl) is the chosen detector in order to provide a very good light output and at the same time a good absorption efficiency for the scattered photons (see the next figure). The energy lost in collisions with the I atoms at the NaI crystal, results in the emission of luminescence light photons (in the visible wavelength range, UV range preferentially) from the excited Tl atoms which are intentionally incorporated into the NaI crystal. But as the scintillating material is not transparent to its own emitted light, a wave shifter is also used. The wave shifter is a material that absorbs light of one frequency and emits it at another. It is obtained that the amount of emitted light is proportional to the energy deposited by the photon in the scintillator [11], [12], [13], [17], [46].

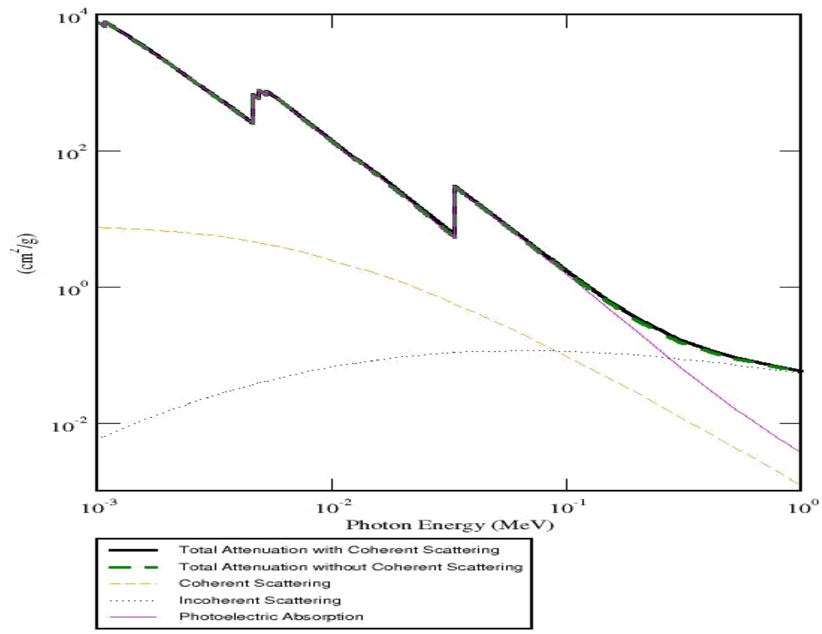

Fig.29. Cross sections of sodium iodide with 1.7% Tl by weight, showing the total absorption and fractional components due to Compton scattering, photoelectric absorption and coherent scattering [46]

The light signal is transformed into an electric signal when the light photons strike the photocathode of the photomultiplier tube (PMT) liberating electrons via photoelectric effect.

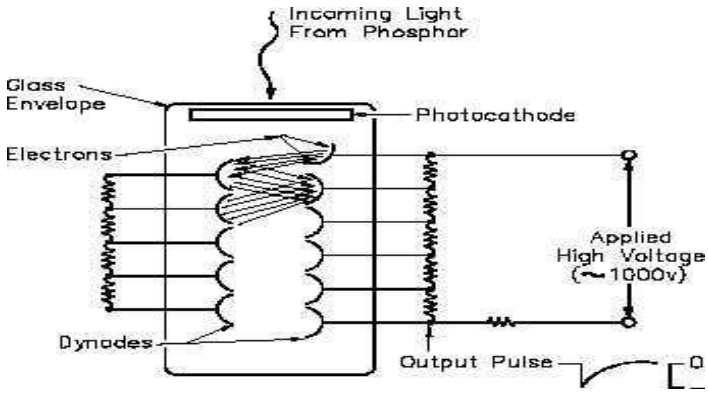

Fig.30. The photomultiplier tube [46]

The PMT provides the amplification of the signal in the following way: when the electrons are accelerated through it, they produce a cascade of electrons each time they impact at the several electrodes (called dynodes) placed inside. This process is repeated resulting in an "avalanche" effect. It results in a detectable signal and amplified signal at the anode (output) of the PMT. An amplification of the original signal by a factor of 10^5 to 10^9 at the anode is typical. The total charge is given by $Q_o = \eta N e_o$, where η (can be from 3 to 5) is the gain per stage, N (can be from 10 to 14) is the number of stages and e_o is the electron charge. The next figure illustrates the photomultiplier tube [11], [12], [13], [17], [46].

The preamplifier converts the pulse from the photomultiplier anode (a charge pulse) to a voltage pulse using a capacitor. This voltage pulse has a peak amplitude which is proportional to the energy absorbed from the incident radiation [11], [12], [13], [17], [46].

After the preamplifier, the signal is further amplified by a main amplifier and shaped to obtain either optimum energy resolution or time resolution. Then, the pulse height (amplitude) is measured and sorted by a computer using a multi-channel analyser (MCA) board [46]. If one irradiates the NaI detector with monoenergetic photons and analyses the scintillation light with the MCA and the oscilloscope, it is possible to see three peaks in a typical gamma spectrum as it is showed in the next figure.

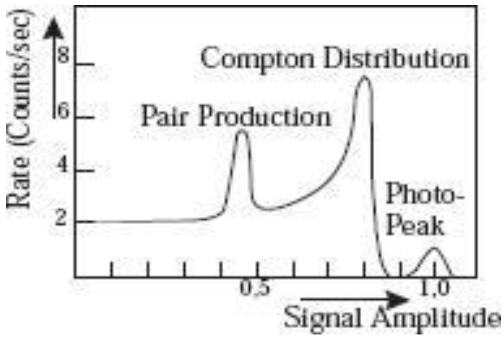

Fig.31. Typical gamma spectrum in a NaI(Tl) crystal if only primary processes occur [46]

The Compton edge preceded by a wide Compton distribution is the peak in the middle. The photo peak and pair production are the other two peaks. The photon transfers the energy to the electron totally (by Photo-effect or pair production) or partially (Compton effect). In this experiment, it is used the photo peak for a quantitative measurement.

2.3. Experimental Procedure

The main amplifier amplifies the signals of the preamplifier. The signals are adapted for the registration in the multichannel analyzer (amplitude has to be about +8 V for 1000 Ω of the input impedance). Also, the main amplifier supplies pulses of nearly gaussian shape (2-8 µs) which are optimum for the ADC (analog to digital converter) in the MCA (multichannel analyzer). It shortens the decay time of the pulse to prevent the mixing of two pulses (pile up effect).

The ADC in the MCA transforms the amplitude of the analogic signal to a digital number, which is interpreted as an address in the memory of the MCA. The next time of a signal with this amplitude (address) is received with the content of this address but incremented by +1. So, the MCA is like a number counter, which is triggered by the ADC. The spectrum is the counting rate as a function of the channel number [11], [12], [13], [17], [46].

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2.4. Results and Data Analysis

Calibration of the MCA

We check the spectra for different sources without Al-pole. We identify the channel numbers and match them with the corresponding energy for each source. For this calibration, we use the following gamma ray sources: Cs 137 (661.64 keV), Ba 133 (303 keV and 356 keV), Na 22 (511 keV). The spectrum for Cs137, Ba133 and Na22 can be seen in the next figures respectively. In the table 10, the channel number and the corresponding peak energy for every source are listed [46].

Table 10. The channel number and the corresponding peak energy for every peak of Ba, Na and Cs sources

Source	Y(channel)	X(Energy keV)
Ba133	374	303
Ba133	438	356
Na22	624	511
Cs137	796	662

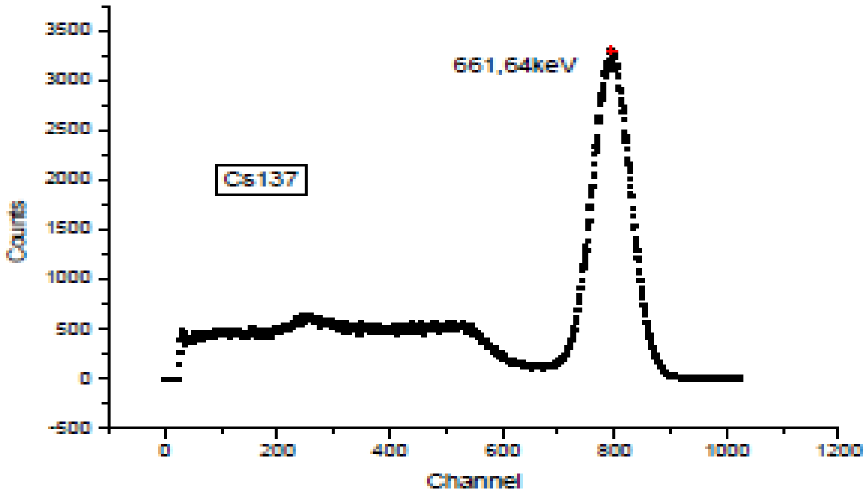

Fig.32. Gamma spectrum for Cs 137

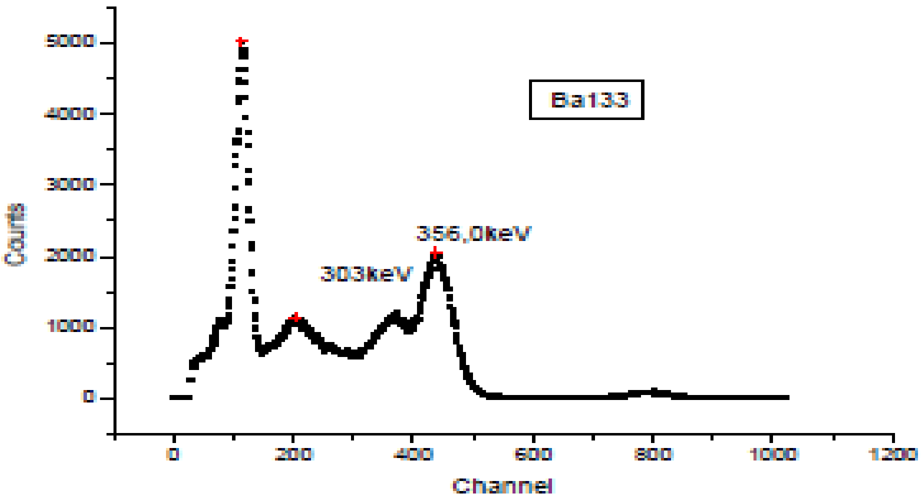

Fig.33. Gamma spectrum for Ba 133

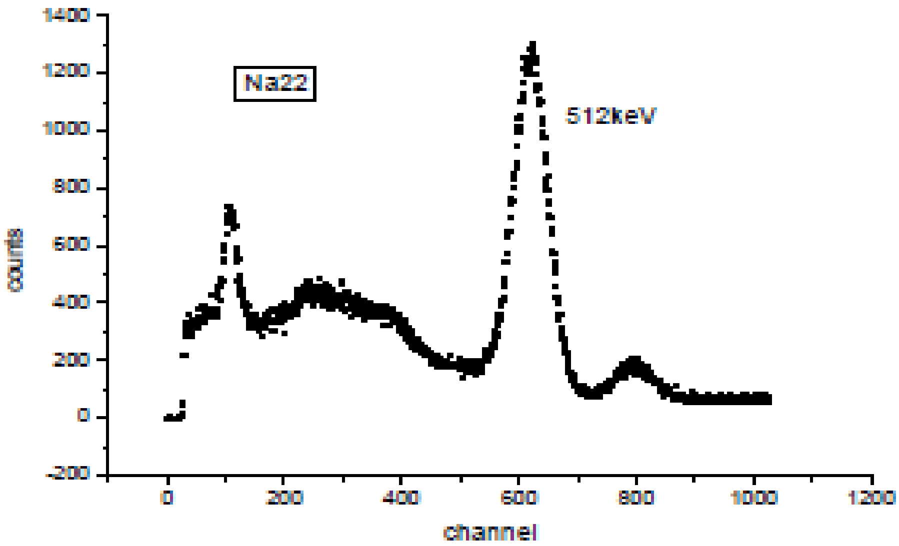

Fig.34. Gamma spectrum for Na 22

We can do a plot of the energy vs the channel number and then, a linear fit to it (see the next figure). It is obtained a linear relationship of the form: Y = A + BX where: Y is the energy in keV, X is the channel number and A and B are constants. In our case, the linear fit obtained is:

Y=-16.01+0.85X A=-16.01 B=0.85

The errors of A and B are:

$\sigma_A = \Delta A = 4.73$ and $\sigma_B = \Delta B = 0.01$

Therefore, the linear equation can be written as the following way:

$Y = -(16.01 \pm 4.73) + (0.85 \pm 0.01)X$

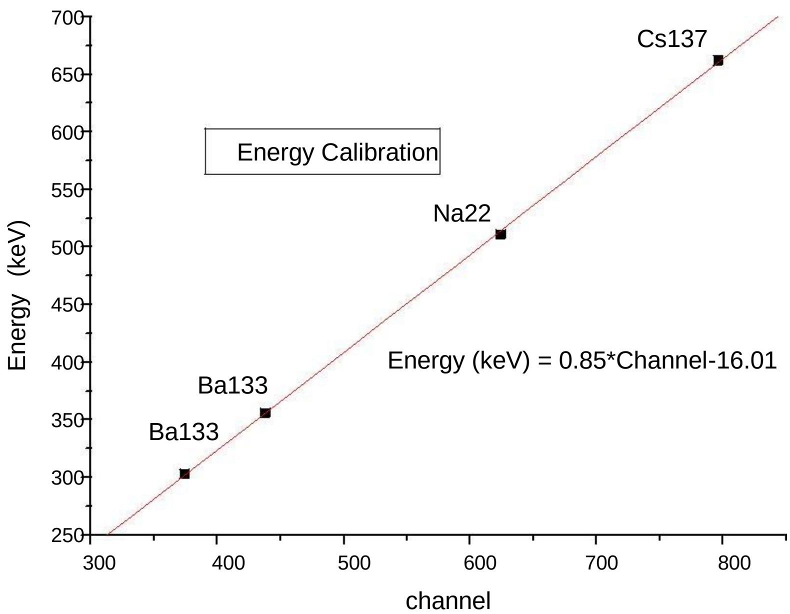

Fig.35. Energy-Channel relationship obtained from the data table 10

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2.5. Measurement of the energy dependence of the Compton scattered photons

Photons with an energy E_γ are scattered at the aluminum target with an angle Φ and the resulting energy E_γ' after the scattering is given by:

$$E'_\gamma = \frac{E_\gamma}{1+\frac{E_\gamma}{m_o c^2}(1-\cos\Phi)}$$ **Energy of the scattered photon**

Then, the angle is changed, and a gamma spectrum is taken each time.

Therefore, the energy of the scattered photon can be obtained, and it is possible to compare the result with the theoretical prediction given by the equation. For different angles between 0° and 104.28°, we have obtained the spectra shown in the next figures.

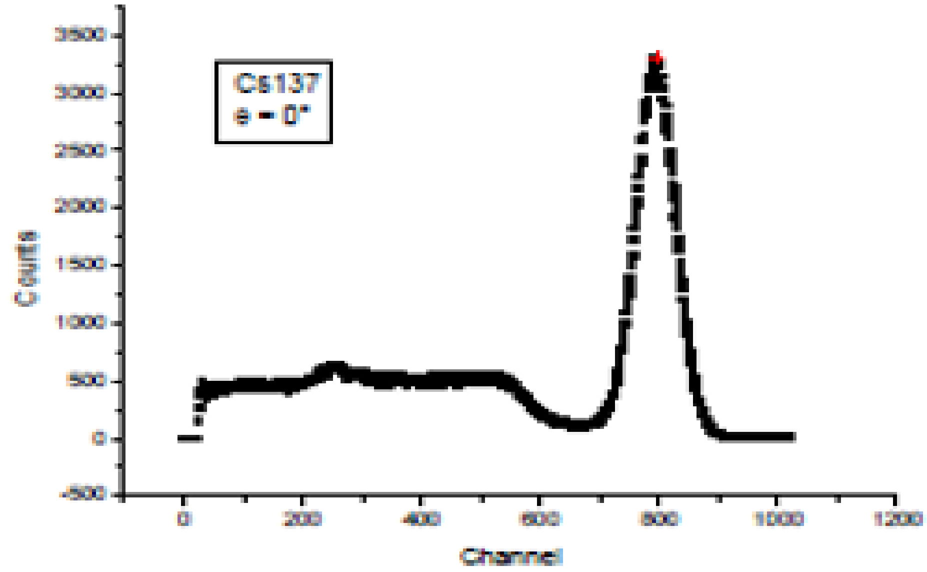

Fig.36. Cs 137 gamma spectrum for a scattering angle of 0^o

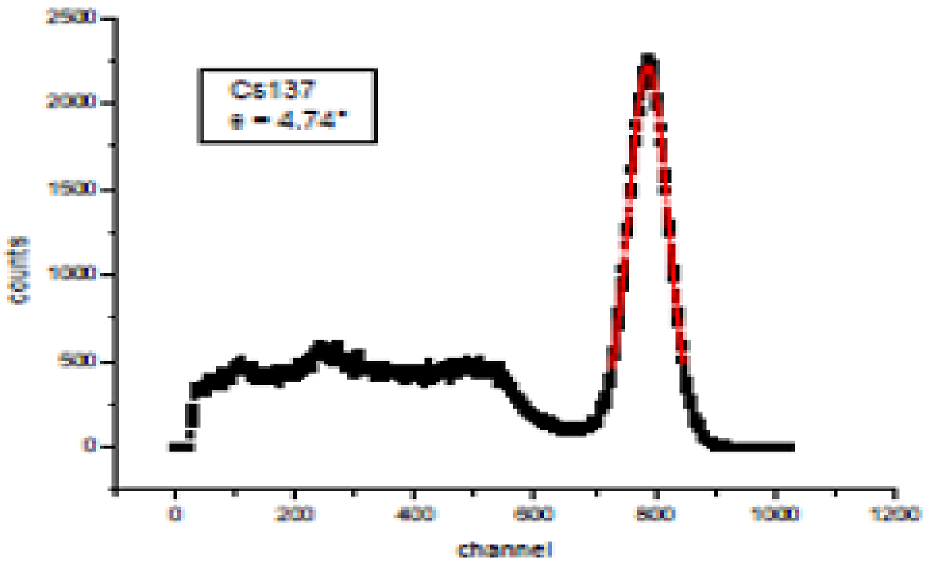

Fig.37. Cs 137 gamma spectrum for a scattering angle of 4.74^o

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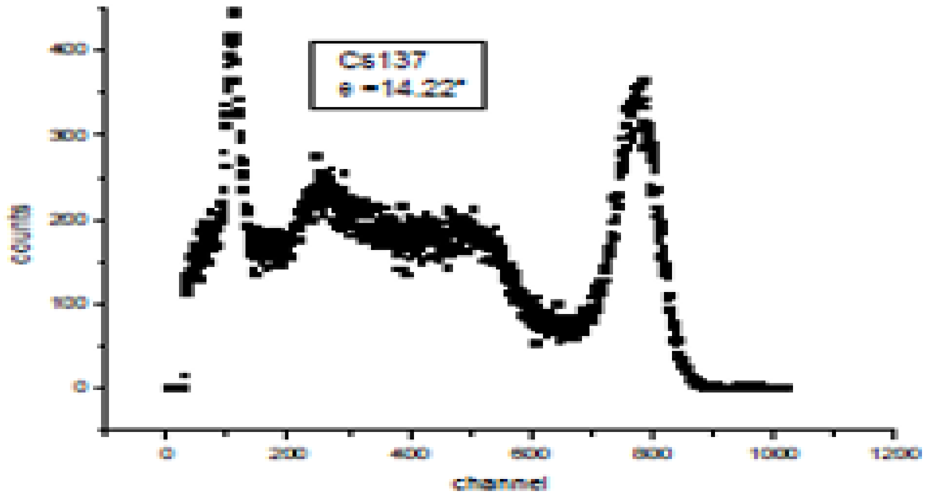

Fig.38. Cs 137 gamma spectrum for a scattering angle of 14.22°

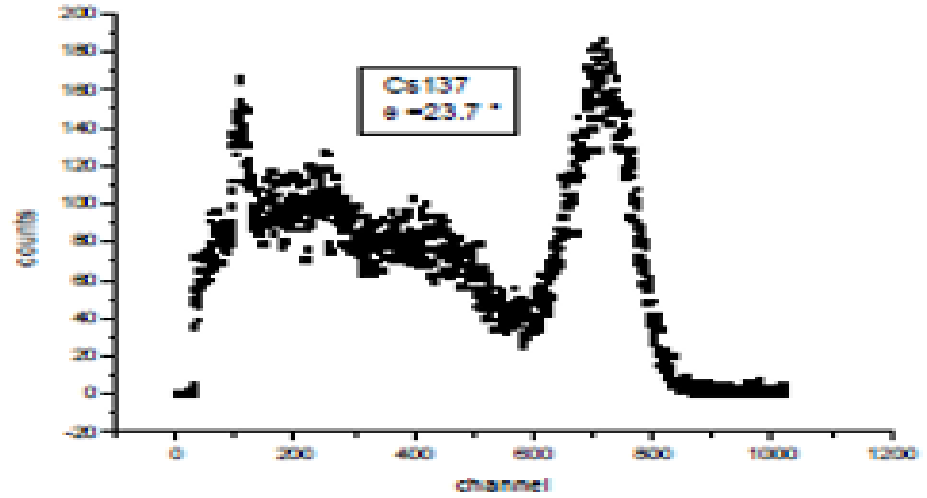

Fig.39. Cs 137 gamma spectrum for a scattering angle of 23.7°

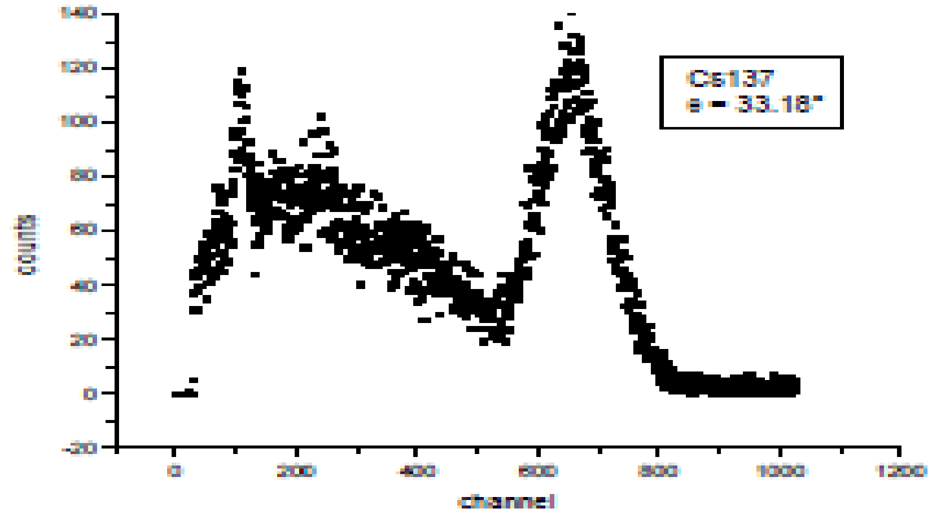

Fig.40. Cs 137 gamma spectrum for a scattering angle of 33.18°

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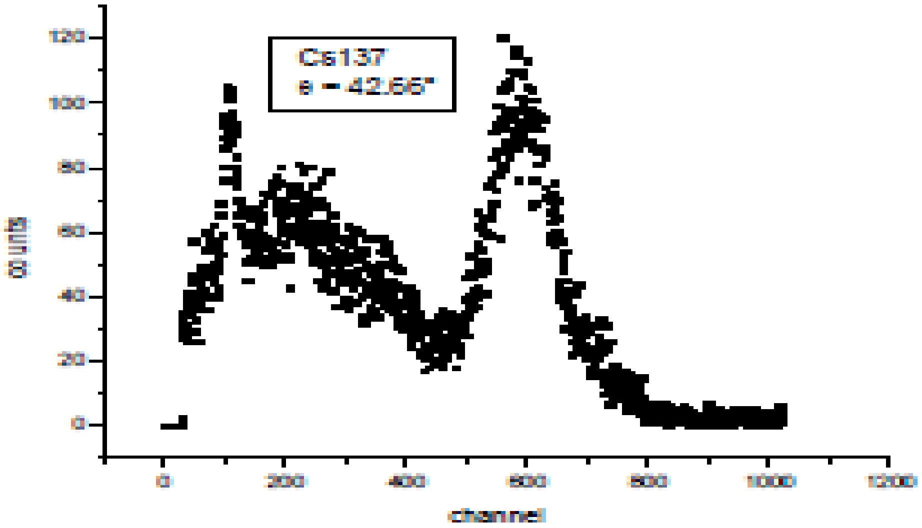

Fig.41. Cs 137 gamma spectrum for a scattering angle of 42.66°

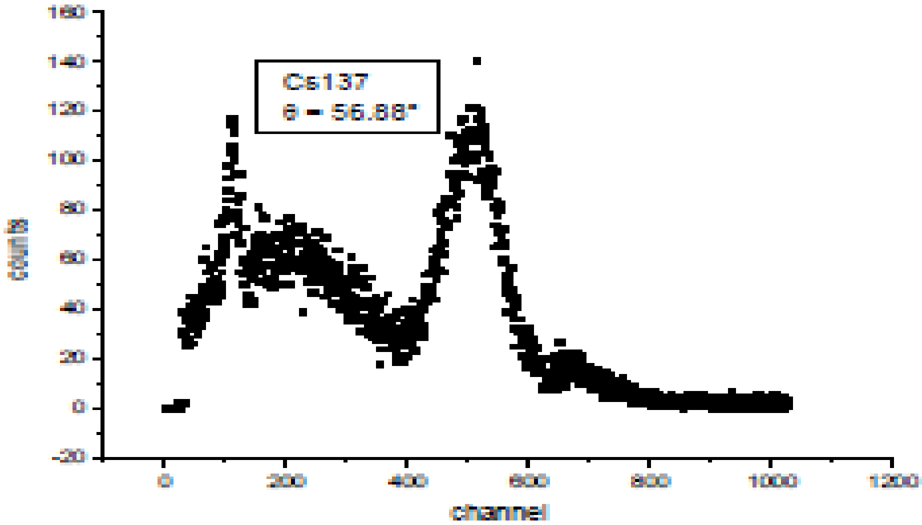

Fig.42. Cs 137 gamma spectrum for a scattering angle of 56.88°

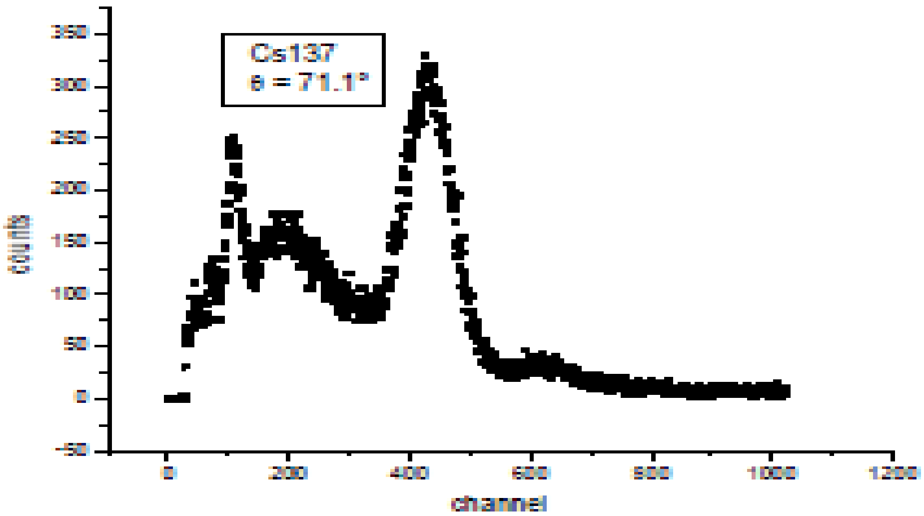

Fig.43. Cs 137 gamma spectrum for a scattering angle of 71.1°

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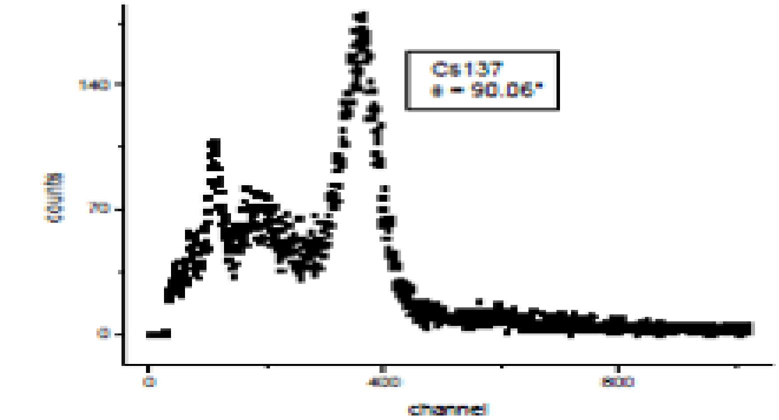

Fig.44. Cs 137 gamma spectrum for a scattering angle of 90.06°

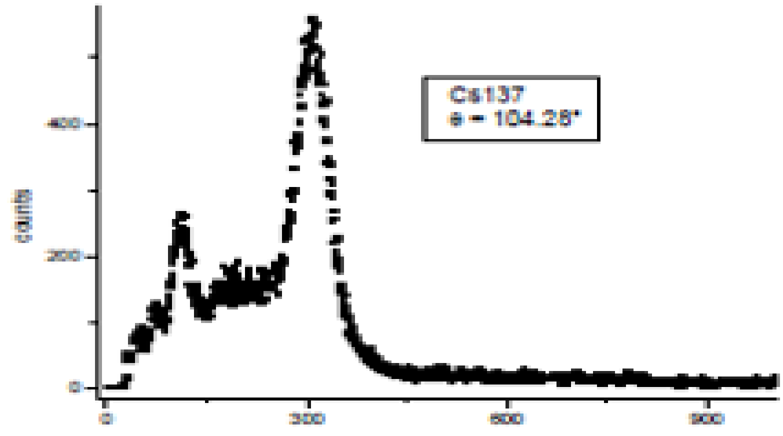

Fig.45. Cs 137 gamma spectrum for a scattering angle of 104.28°

We need to find the peak channel number from the spectra in order to determine the energies of the scattered photons. The next table shows the peak channel, energy channel with the respective errors and with an error of 0.5 in the peak determination for all peaks.

Table 11. The peak and the energy of the scattered gamma photon for different scattering angles, the error calculation for the energy of the scattered photon is done according to error propagation

Angle (deg.)	Peak (channels)	Error Peak (channels)	E´exp. (keV)	$E´_{theo.}$ (keV)	Error $E´_{exp.}$ (keV)	Error $E´_{theo.}$(keV)
4,74	789	0,5	654,64	658,725936	12,194467	1,20322243
14,22	777	0,5	644,44	636,416931	12,1171707	3,33874756
23,7	720	0,5	595,99	596,560875	11,7598012	4,80061798
33,18	657	0,5	542,44	546,402927	11,3849824	5,48381627
42,66	573	0,5	471,04	492,919836	10,921805	5,5265315
56,88	512	0,5	419,19	416,970102	10,6144866	4,88852174
71,1	438	0,5	356,29	352,946674	10,278002	3,95727651
90,06	364	0,5	293,39	288,281391	9,98511517	2,79129367
104,28	318	0,5	254,29	253,209744	9,82676574	2,0874741

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The energy of the scattered photon (E'_{exp}) is found using the energy-channel relation which was found in the calibration part: E (Energy keV) =A+B*channel where X=channel and Y=E(Energy) and Y=-16.01+0.85X A=-16.01 B=0.85. The errors of A and B are: $\sigma_A = \Delta A = 4.73$ and $\sigma_B = \Delta B = 0.01$

The expected value of the energy (E'_{theo}) for a given scattering angle is found using formula: $E'_\gamma = \frac{E_\gamma}{1+\frac{E_\gamma}{m_o c^2}(1-\cos\Phi)}$.

The error of the measured energy can be obtained according to the error propagation formula [18]. Therefore, it is obtained:

$$\Delta E_{exp}(\text{keV}) = \sqrt{(\Delta A)^2 + (\text{channel} * \Delta B)^2 + (B * \Delta\text{channel})^2}$$

Where, ΔA and ΔB are just the uncertainties σ_A and σ_B, and Δ channel is the error in the peak given by the gaussian fit. Because the NaI(Tl) detector was rotated manually (to change the scattering angle), there might be an error in the angle measurement. Therefore, we have considered it as a systematic error which can influence the calculation of the theoretical or expected energy (E'_{theo}). Therefore, the error in E'_{theo} is calculated as a propagation error of the scattering angle Φ.

$$\Delta E'_{theo}(\text{keV}) = \left|\frac{dE'}{d\Phi}\right| \Delta\Phi = \frac{E_\gamma{}^2 m_o c^2}{(m_o c^2 + E_\gamma - E_\gamma \cos\Phi)^2} \sin\Phi\, \Delta\Phi$$

We can take ΔΦ as 1° and therefore, in the worst case our apparatus has a mis-calibration of 1°. As it can be seen from the table 11, this error is always smaller than the random error.

At the next figure, it is compared the experimental and the theoretical energies as a function of the scattering angle.

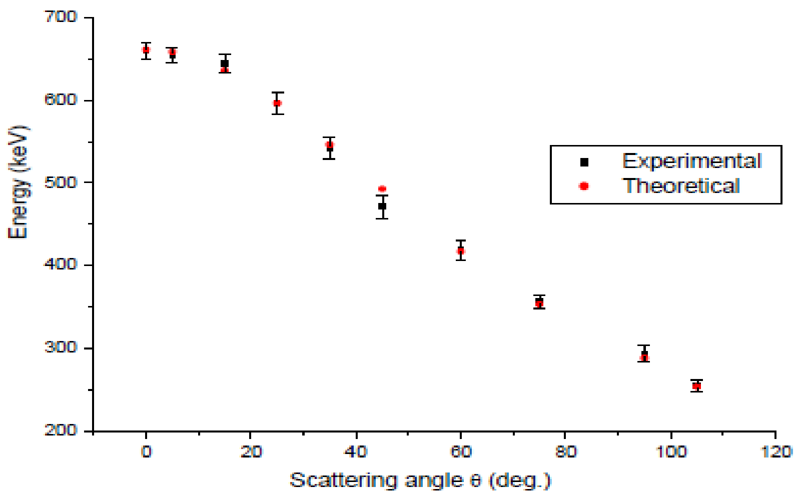

Fig.46. Experimental and theoretical energies of the scattered gamma photons as a function of the scattering angle

It is possible to see that the experimental energy follows the same dependency on the scattering angle as the theoretical. Furthermore, both results are always close to each other and almost the expected result lies within the experimental error.

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By using the equation: $E'_\gamma = \frac{E_\gamma}{1+\frac{E_\gamma}{m_o c^2}(1-\cos\Phi)}$, we can calculate the electron rest mass:

$$m_o c^2 = \frac{E_\gamma E'_\gamma (1-cos\Phi)}{E_\gamma - E'_\gamma}$$

The next table shows the calculated electron´s rest mass for every scattering angle with their corresponding errors. The error coming from the statistics as well as the one considered as systematic in the measured angle are shown at the next table.

Table 12. Electron rest mass calculated from the different energies of the scattered gamma photon

Angle (deg.)	E´ exp. (keV)	Calc. Electron Mass (keV)	Error Mass (keV) *Stat.*	Error Mass (keV) *Syst.*	Total Error (keV)
4,74	654,64	211,407197	374,121732	234,829307	441,714697
14,22	644,44	758,806695	551,635702	72,1885686	556,339049
23,7	595,99	506,071737	101,150791	42,6637101	109,78012
33,18	542,44	490,428333	57,4263056	29,9994018	64,7900045
42,66	471,04	432,261431	34,9696387	22,8734466	41,7860047
56,88	419,19	518,424181	36,0065958	16,4848944	39,600842
71,1	356,29	521,488525	32,7629365	12,4920932	35,0636906
90,06	293,39	527,269939	32,406263	8,9176122	33,6108567
104,28	254,29	514,540961	32,4611931	6,94013492	33,1947967

The statistical error is calculated again from the error propagation. Therefore, we have obtained the next formula:

$$\Delta m_o c^2 (keV) = \left|\frac{d m_o c^2}{d E_\gamma{}'}\right| \Delta E_\gamma{}' = \frac{E_\gamma{}^2 (1-cos\Phi)}{(E_\gamma - E_\gamma{}')^2} \Delta E_\gamma{}'$$

Here, $\Delta E_\gamma{}'$ is the calculated error of the measured energy E'_{exp} shown at the table 12.

Also, we consider an independent error coming from the angle measurement (systematic error). Thus, an error at the mass is given by:

$$\Delta m_o c^2 (keV) = \left|\frac{d m_o c^2}{d\Phi}\right| \Delta\Phi = \frac{E_\gamma E_\gamma{}' \sin\Phi}{E_\gamma - E_\gamma{}'} \Delta\Phi$$

Finally, as both errors are independent, it is possible to add them at quadrature to find the total error:

Total Error (keV)=$\sqrt{(Stat.)^2 + (Syst.)^2}$

Because we have several measurements for the rest mass and each one with its own uncertainty, we can apply the weighted average method to find the best value for the mass and its error.

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The weighted average method establishes that if one has N measurements of the same quantity X, each measurement xi with uncertainty σi, then the best value for X is given according to the following formula:

$x_{best} = \frac{\sum_{i=1}^{N} w_i x_i}{\sum_{i=1}^{N} w_i}$ $\sigma_{best} = \frac{1}{\sqrt{\sum_{i=1}^{N} w_i}}$ Where, $w_i = \frac{1}{\sigma_i{}^2}$: the weight of the measurements.

The final result is given by: $\mathrm{m_o c^2} = (506.28 \pm 15.55)$ keV

We have used the values from 23.7° to 104.28° because the results for small scattering angles (0°, 5° and even 15°) have quite large errors. Also, it is not possible to calculate the electron mass for 0°, since it means that there is no scattering and hence the equation: $\mathrm{E'_\gamma} = \frac{\mathrm{E_\gamma}}{1+\frac{\mathrm{E_\gamma}}{\mathrm{m_o c^2}}(1-\cos\Phi)}$ can not be used and $\mathrm{E'_\gamma} = \mathrm{E_\gamma}$. The expected value for the electron rest mass (511 keV) is within 1 uncertainty of our experimental result.

2.6. Compton Edge

For the determination of the compton edge, we take the spectrum for no scattering. It is for Φ = 0° as we can see at the figure 36. Besides, the compton continuum is clearly identified in this spectrum, nonetheless the compton edge is not. It is rather smeared and for this reason we make a zoom as it is shown in the next figure.

It is possible to divide the smearing part into two sections and we can say that the Compton edge is approximately there. In our case, we get a Compton edge located around channels 580 and 590. By taking the channel 585 as a representative value, we can calculate the energy corresponding to this channel by using again our energy-channel relation: E(Energy keV)=A+B*channel A= -16.01 B= 0.85

Energy Compton Edge (keV) (Y) = - 16.01+0.85 * Channel (X)

Energy Compton Edge = - 16.01+0.85*585 = 481.24keV

By taking a conservative uncertainty at the channels, we can consider Δch = 5 channels and we calculate the error as follow: $\Delta \mathrm{E_{exp}}(\mathrm{keV}) = \sqrt{(\Delta \mathrm{A})^2 + (\mathrm{channel} * \Delta \mathrm{B})^2 + (\mathrm{B} * \Delta \mathrm{channel})^2}$, σ_A =ΔA= 4.73 and σ_B =ΔB= 0.01 and $\Delta \mathrm{E_{exp}}(\mathrm{keV}) = \sqrt{(4.73)^2 + (585 * 0.01)^2 + (0.85 * 5)^2}$ =8.64 keV

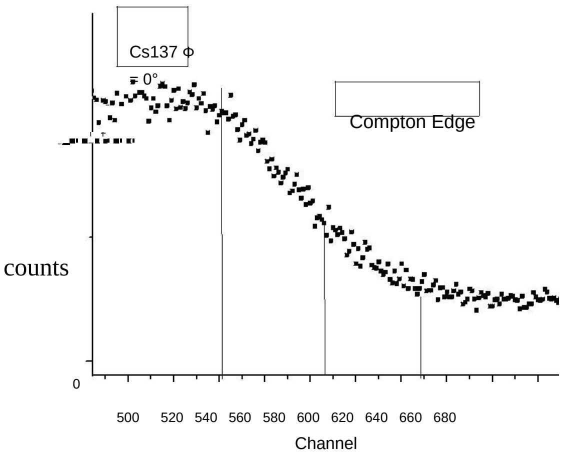

Fig.47. Zoom of the Cs137 spectrum from the figure 36 to identify the Compton edge

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Whereas the theoretical Compton edge for the 661.64 gamma emitted from Cs137 would yield: E_γ=661.64 keV

$K = E_\gamma - E'_\gamma = \frac{E_\gamma^2(1-\cos\Phi)}{m_o c^2 + E_\gamma(1-\cos\Phi)}$ Kinetic energy of the electron

Compton Edge=$K_{emax} = \frac{2E_\gamma^2}{m_o c^2 + 2E_\gamma}$ Maximum Kinetic energy of the electron ($\Phi=180^o$: backscattering of the photon)

K_{emax}=477.32 keV

By taking a conservative approach in the determination of the channel number, we can see that the experimental and expected values agree very well.

2.7. Compton Scattering Cross Section

The Klein-Nishina cross section is given by the formula:

$$\frac{d\sigma_c}{d\Omega} = \frac{r_o^2}{2}\frac{1}{[1+\alpha(1-\cos\Phi)]^2}\left[1+\cos^2\Phi + \frac{\alpha^2(1-\cos\Phi)^2}{1+\alpha(1-\cos\Phi)}\right] \qquad \alpha = E_\gamma/(m_o c^2)$$

Φ: the scattering angle of the photon

r_o=2.82 fm is the classical electron radius

m_o: the mass of the electron.

The value $d\sigma_c/d\Omega$ is the scattering photon into the solid angle defined by $d\Omega=2\pi\sin\Phi d\Phi$.

The experimental formula can be obtained according to:

$$\frac{d\sigma}{d\Omega}(\Phi) = \frac{N_c(\Phi)}{\Delta\Omega N N_o}\,\frac{cm^2}{ster.electron}$$

$\Delta\Omega$: Geometry factor of the detector.

N: Number of target electrons.

N_0: the flux of the incident photons on the detector.

Where, $N_c(\Phi) = \frac{I_c(\Phi)}{e(E)g(E)} = \frac{1}{e(E)g(E)}\frac{I(\Phi)-(IL(\Phi)+IR(\Phi))}{time}\,\frac{phot.}{sec}$

$I(\Phi)$: Sum of the photons in the photopeak or just the area below the photopeak.

$IL(\Phi)$ and $IR(\Phi)$: Sum of the photons in half the photopeak interval to the left and right of the photopeak, respectively. They are also calculated as the area of such intervals.

Time: the measuring time of the spectra.

$N_0=I_0$/Area irradiated in the Al pole and I_0 is calculated in the same way as $I(\Phi)$ but without the pole.

The factors e(E) and g(E) are the efficiency from the detector and the peak to total ratio, respectively.

We calculate each of the factors in the following way:

- The geometry factor $\Delta\Omega$ is just the detecting area of the NaI(Tl) divided by the distance between the Al pole and the detector squared.

  In this case: $\Delta\Omega = 15.8\text{cm}^2/(11.5\text{cm})^2 = 0.12$

- $N = V_o \rho \frac{L}{A} Z = \frac{(5.6\ cm^3)(2.72\frac{g}{cm^3})(6.02 * 10^{23} Mol^{-1})(13)}{27} = 4.4 * 10^{24} electr.$

- $I(\Phi)$ is obtained by doing a fit to the background peak corrected. The background correction is carried out by subtracting with a straight line from the spectrum in the interval that spans the peak. The figure 48 depicts this procedure. Also, the area of the two intervals beside the peak are found out by a fitting program in order to find the $N_c(\Phi)$ like the above formula establishes. The values measured for the time and the areas $I(\Phi)$, $IL(\Phi)$ and $IR(\Phi)$ are listed at the table 13.

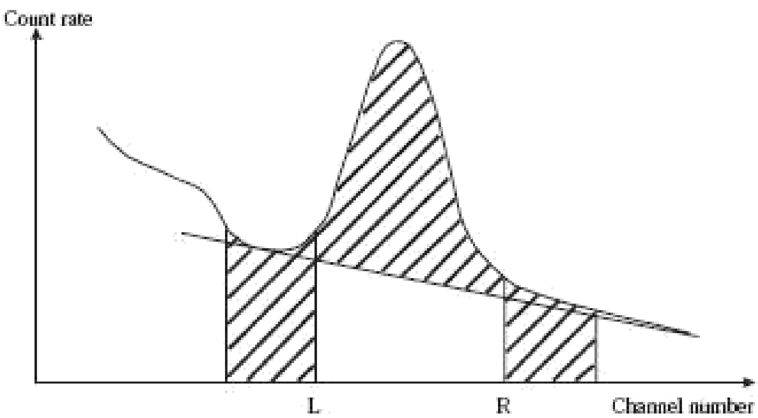

Fig.48. The area around the photopeak is divided into three intervals. The interval (L-R) contains the main part of the photopeak, and the limits are not strictly fixed. The other intervals have the half of the width (L-R) [46]

Table 13. Time and Area measurements from the peak and the left and right intervals beside it for every scattering angle

Angle (deg.)	Time (s)	Dead time (%)	Area L (phot)	Area R (phot)	Area Peak (phot)	Error Area Peak (phot)
0	39	12	12164	2479,5	250682,88	1446,78
4,74	38	9	11137	1098,5	167935,33	1246,81
14,22	100	1	5992	311	26221,22	458,1
23,7	310	0	5162	378	19108,82	584,4
33,18	294	0	3838	535	13719,04	422,85
42,66	297	0	3519,5	1314	11266,77	349,84
56,88	365	0	3095,5	1440	10246,05	422,1
71,1	1011	0	6730	2617,5	22528,44	702,77
90,06	480	0	3022,5	1023	10425,68	235,63
104,28	1090	0	7465	3049	25975,65	498
0 (No Pole)	18	16	8211	3081,5	151511,77	1245,74

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The factors e(E) and g(E) can be obtained from the graphs at the figures 49 and 50.

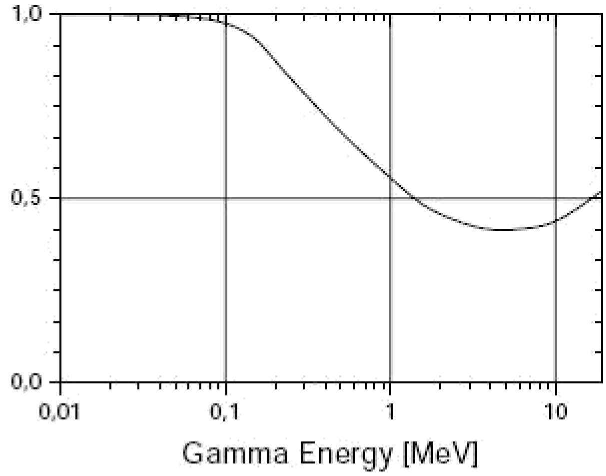

Fig.49. Detector efficiency as a function of the energy of the gamma photon [46]

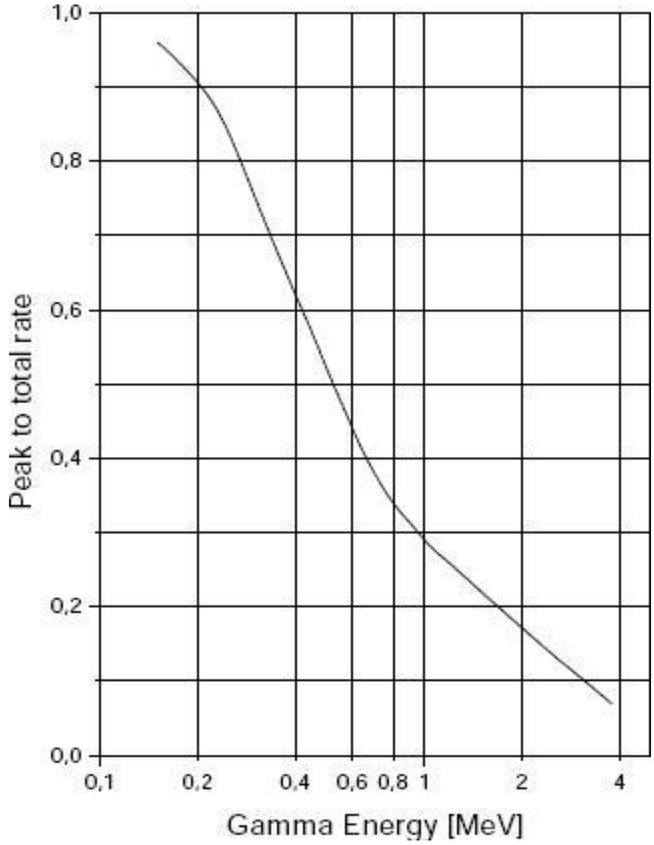

Fig.50. Peak to Total Rate, as a function of the energy of the gamma photon [46]

The factor e(E) is due to the detector efficiency. Besides, we get different values of this factor for each energy of the scattered photon which impinges into the detector. We have values in a range from e(E) = 0.55 for $\Phi = 0°$ up to e(E) = 0.78 for $\Phi = 104.28°$. For other hand, g(E) = 0.6 is just the one given for the energy of the incoming gamma photon from the source with an energy of 0.661MeV. (Notice that the Peak to Total Rate is generally given as the fraction of the photoevents divided by the total number of events. Therefore, if we want to consider a similar ratio but for the Compton events, we subtract the value at the figure 50 from 1 which is 0.4.)

We calculate the cross section with all the necessary values, and we obtain the results listed in the table 14. The errors listed at the table 14 are calculated using the error propagation method but this time for the peak area I(Φ) and the photon flux N_0. It is given by the next formula:

$$\Delta\frac{d\sigma}{d\Omega} = \sqrt{[\frac{d\left(\frac{d\Phi}{d\Omega}\right)}{dI(\Phi)}\Delta I(\Phi)]^2 + [\frac{d(\frac{d\Phi}{d\Omega})}{dN_o}\Delta N_o]^2}$$

Table 14. Cross Sections for the different angles with their corresponding errors

Angle Φ (deg.)	Cross Section 2 (cm^2 /ster-electr)	Error Cross Section I(Φ) part (cm^2/ster-electr)	Error Cross Section N_0 Part (cm^2/ster-electr)	Total Error 2 (cm^2 /ster-electr)
0	1,87278E-23	1,00236E-25	6,14813E-26	1,17589E-25
4,74	1,18304E-23	8,27241E-26	3,8546E-26	9,12637E-26
14,22	5,2863E-25	1,06165E-26	6,01618E-28	1,06335E-26
23,7	1,11107E-25	4,17857E-27	4,03815E-29	4,17877E-27
33,18	7,93489E-26	3,13487E-27	3,04087E-29	3,13502E-27
42,66	5,23233E-26	2,48458E-27	1,98492E-29	2,48466E-27
56,88	3,49721E-26	2,25725E-27	1,07953E-29	2,25727E-27
71,1	2,7501E-26	1,28037E-27	3,06479E-30	1,28037E-27
90,06	2,61933E-26	8,44709E-28	6,14826E-30	8,44732E-28
104,28	2,72362E-26	7,66019E-28	2,81529E-30	7,66025E-28

For the angles shown in the above graph, we can see that the results have the same order of magnitude as the theoretical curve and that the shapes are similar. We have not included at the plot the values for 0°, 4.74° and 14.22° because their order of magnitude is 1 to 2 times bigger than the theoretical values.

Nevertheless, the results for the cross-section don´t lie exactly within the predicted curve of the Klein-Nishina equation. One possible reason for the discrepancy between the expected and the measured results is that the irradiated volume of the pole cannot be considered as a point-like source of compton scattered photons. It means that there is a probability that the scattered photon may interact many times inside the pole before reaching the detector. This eventuality is not considered in the formula that we have used. Another reason is the uncertainty that there might be in the determination of the flux of photons at the target, which was found out by following geometrical considerations.

2.8. Determination of Δch, ΔΦ, and ΔE´, Rest Mass

Δch: If we want to determine the error in the channel, we need to perform two times the same energy measurement with the same conditions (for example the calibration and the experimental) and the difference in the peak location

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channel number will give the channel error. If we want to be more precise about this error, it is better to perform one measurement more at the same conditions and to apply statistical methods.

ΔΦ: We measure the radius of the NaI(Tl) scintillator detector (r) and the distance between the detector and the Al target (d). Then, we obtain the next formula by applying geometrical considerations: tan ΔΦ= r/d, therefore we can obtain ΔΦ.

ΔE´: The energy resolution (energy error) can be obtained by measuring the width of the peak. It is a different way of error propagation to calculate the energy error.

We fit a Gaussian shape curve to the peak and then we can find σ (energy error). σ≈0.42*fwhm, where fwhm is the full width at half maximum of the Gaussian. Finally, we can convert σ to units of energy by using the energy-channel calibration equation. The formula of the Gaussian curve is given by the next formula:

$$C = C_o e^{-\frac{(N-\mu)^2}{2\sigma^2}}$$

Where, C is the number of counts, C_o is the number of counts at the peak of the distribution, N is the channel number, μ is the channel number where the peak is located, and σ is the width of the distribution.

Rest Mass: Other way different of error propagation to calculate the rest mass of the electron is by doing a plot of 1/E´ versus (1-cosΦ). Then, we can apply the least square method to fit the best line to the points. Afterward, we can calculate the slope and find the rest mass. If we analyze the formula: $E'_\gamma = \frac{E_\gamma}{1+\frac{E_\gamma}{m_o c^2}(1-\cos\Phi)}$

$$\frac{1}{E_\gamma{}'} = \frac{1}{E_\gamma} + \frac{1}{m_o c^2}(1 - cos\Phi)$$

The slope will be given by $1/m_o c^2$. It is possible to find the error of the slope given by the same least square method and by applying the error propagation method to the slope=$1/m_o c^2$.

2.9. Conclusions

The compton scattered energies measured agree very well with the predicted values from the theory. By calculating the electron rest mass, it is quite far with the expected result. Therefore, if we have more measurements for the rest mass, we can improve our result. Also, we estimate the statistical and systematic errors for the measurements. The results suggest that the random statistical errors dominate. If we want to improve the result, we must have more data for smaller angle intervals.

However, as the systematic error implies, there is a limit where taking more data has no effect at the final result. Therefore, it is the setup and the technique which determine the precision of the measurement. Besides, we measure the Compton edge for the Cs137 spectrum with a good agreement with the theory by taking a conservative choice in the channel determination error. At the energy considered (661MeV), we observe that the NaI(Tl) detector detects more photons per unit time when it makes a small angle with respect to the axes determined by the source and the Al pole. It tells us that the scattered photons have a predominant direction of scattering (small angles~5°) for this energy as predicted by the Klein Nishina formula. For other hand, it serves to check that the

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Klein-Nishina formula does represent the Compton scattering cross section.

3. Variant Mass of the Electron at the atom

3.1. An electron orbiting the nucleus

The mass of the electron is m and the mass of the nucleus is M. During the first research about the atom, the Rutherford Model was represented as the electron moves around the nucleus in a circular motion with radius R. After, the motion of the electron was represented as an ellipse as the Sommerfeld Model has established [19], [20], [21], [22], [23], [24]. The nucleus is at the focus of the ellipse as the sun in one focus at the Planetary System [19], [20], [21], [22], [23], [25], [26]. There is a strong analogy between the Planetary System and the Atom System at this model.

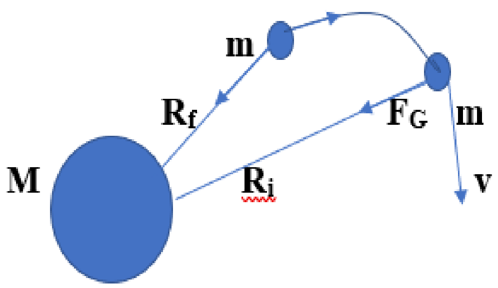

Fig.51. An electron orbiting the nucleus due the Electric Force

The electrical force between the electron and the nucleus gives out the Electrical Potential Energy. At the Planetary System, the gravitational force between the planets and the sun gives out the Gravitational Potential Energy. Respect to the gravitational energy emission, it is demonstrated by theory, experiment and result the discovery formula which describe exactly the variant mass of a particle which emits gravitational energy which was demonstrated by myself at the article: The Fundament of the Mass and Effects of the Gravitation on a Particle and Light in the mass, time, distance, velocity, frequency, wavelength: Variant Mass for a Particle which emits Gravitational Energy for a particle orbiting a large Planet or Sun and for a Binary Star and Variant Frequency for the Light passing close a Gravitational Field from a Massive Object (Sun). The results of the mass formula are of great relevance for Gravitational Interactions. It is in accordance with the classic result for the emission of the total gravitational energy (bond total energy) for a particle orbiting a large Planet or Sun and for a Binary Star.

At the atom, the electron only emits electromagnetic energy or photons at the jump from one stationary orbit to another stationary orbit at the atom. Then, the mass of the electron decreases due the emission of the electromagnetic energy. As result of it, the electron changes its stationary orbit by decreasing the radius R with the nucleus. Afterwards, the electron starts to move in a circular or elliptical motion (rotational motion) around the nucleus due the initial velocity of the electron but without the emission of electromagnetic energy.

3.2. Formula development of the total mass of the electron at the electric potential of the nucleus and quantization formula

$c^2 dm = \frac{dp}{dt} ds + \frac{ke^2}{R^2} dR \qquad c^2 dm = v dp + \frac{ke^2}{R^2} dR \quad v = ds/dt$

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p=mv   m=p/v   The electrical force is equal to the centrifugal force:

$$\frac{ke^2}{R^2} = m\frac{v^2}{R} \qquad \frac{ke^2}{R^2} = \frac{pv}{R}$$

$$\frac{ke^2}{R} = pv \qquad -\frac{ke^2}{R^2}dR = pdv + vdp$$

$$\frac{ke^2}{R^2}dR = -pdv - vdp \qquad c^2d(\frac{p}{v}) = vdp - pdv - vdp$$

$$c^2d(\frac{p}{v}) = -pdv \qquad c^2\frac{vdp-pdv}{v^2} = -pdv$$

$$-c^2\frac{dp}{v} = p(1-\frac{c^2}{v^2})dv \qquad -c^2\frac{dp}{p} = v(1-\frac{c^2}{v^2})dv$$

$$\ln\frac{p}{p_o} = -((\frac{v^2-v_o{}^2}{2c^2}) - (\ln v - \ln v_o))$$

$$\frac{p}{p_o} = \frac{v}{v_o}e^{-(\frac{v^2-v_o{}^2}{2c^2})} \qquad p=mv \quad p_o=m_ov_o$$

$$\frac{mv}{m_ov_o} = \frac{v}{v_o}e^{-(\frac{v^2-v_o{}^2}{2c^2})}$$

$m = m_oe^{-(\frac{v^2-v_o{}^2}{2c^2})}$ total relativistic mass of the electron at the atom

$\Delta m=m_oc^2 - mc^2 = m_oc^2(1-e^{-\left(\frac{v^2-v_o{}^2}{2c^2}\right)})$ total decrease mass of the electron at the atom

If $v^2<<c^2$, then it is obtained for the classical case:

$$e^{-(\frac{v^2-v_o{}^2}{2c^2})} = 1-\left(\frac{v^2-v_o{}^2}{2c^2}\right)+\frac{\left(\frac{v^2-v_o{}^2}{2c^2}\right)^2}{2!}+\cdots$$

$\frac{\left(\frac{v^2-v_o{}^2}{2c^2}\right)^2}{2!}$ and other terms are neglected

$$\frac{ke^2}{R^2} = m\frac{v^2}{R} \qquad \frac{ke^2}{mR} = v^2 \qquad m = m_o(1-\left(\frac{v^2-v_o{}^2}{2c^2}\right))$$

$$\left(\frac{v^2-v_o{}^2}{2c^2}\right) = \frac{ke^2}{2m_oc^2}(\frac{1}{R}-\frac{1}{R_o}) \qquad R_f=R \quad R_i=R_o$$

$m = m_o(1-\frac{ke^2}{2m_oc^2}(\frac{1}{R}-\frac{1}{R_o}))$ mass of the electron: classical approach

$m_oc^2 - mc^2 = \frac{ke^2}{2}(\frac{1}{R}-\frac{1}{R_o})$ total decrease mass: classical approach

$m_oc^2 - mc^2 = \frac{ke^2}{2R}$ total decrease mass of the electron: classical approach (R_o->∞)

It is in accordance with the total energy for the electron (bound energy) at the atom for the classical approach which is equal to the decrease mass of the electron and equal to the electromagnetic radiation emitted by the electron at the atom. The electron at the atom only can take restricted positions which are explained by quantum mechanics.

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For the total energy of the electron or bound energy, it is obtained:

$E = mc^2 - m_o c^2 \qquad m = m_o e^{-(\frac{v^2 - v_o^2}{2c^2})}$

$E = m_o c^2 (e^{-\left(\frac{v^2 - v_o^2}{2c^2}\right)} - 1)$ total relativistic energy for the electron in the atom

If $v^2 << c^2$, then it is obtained for the classic case:

$$e^{-(\frac{v^2 - v_o^2}{2c^2})} = 1 - \left(\frac{v^2 - v_o^2}{2c^2}\right) + \frac{\left(\frac{v^2 - v_o^2}{2c^2}\right)^2}{2!} + \cdots$$

$\frac{\left(\frac{v^2 - v_o^2}{2c^2}\right)^2}{2!}$ and other terms are neglected

$E = -m_o\left(\frac{v^2 - v_o^2}{2}\right)$ classical approach

$\frac{ke^2}{R^2} = m\frac{v^2}{R} \qquad \frac{ke^2}{mR} = v^2 \qquad \left(\frac{v^2 - v_o^2}{2}\right) = \frac{ke^2}{2m_o}(\frac{1}{R} - \frac{1}{R_o})$

$E = -m_o\left(\frac{ke^2}{2m_o}(\frac{1}{R} - \frac{1}{R_o})\right) \qquad E = -\left(\frac{ke^2}{2}(\frac{1}{R} - \frac{1}{R_o})\right)$

$E = -\left(\frac{ke^2}{2R}\right)$ total energy for the electron at the atom or bound energy

The kinetic energy for the electron at the atom for the classical approach is the same formula but in absolute value:

$K = \left(\frac{ke^2}{2R}\right)$

The total lost mass energy of the electron at the atom is given by the formula demonstrated and it is equal in absolute value to the total bound energy of the system which is emitted as electromagnetic energy when the electron does the transition from one orbit to another orbit with fewer radiuses. The emission of the total electromagnetic energy E produces a decrease mass of the particle: $\Delta m = E/c^2$.

If $R_f < R_i$ $\Delta E = E_f - E_i = (-\frac{ke^2}{2R_f}) - \left(-\frac{ke^2}{2R_i}\right)$ is negative, it is given out energy which means that the electron losses mass, the electron increases the kinetic energy and the velocity but decreases the electrical potential energy (more negative). Part of the lost energy (electromagnetic energy emission or photon) is given out by decreasing the potential energy and part by increasing the kinetic energy.

If $R_f > R_i$ ΔE is positive, which means that additional energy is given to the electron or additional work is done on the system, the electron decreases the kinetic energy and the velocity but increases the potential energy (less negative). Part of the work or additional energy is used to increase the potential energy and part to diminish the kinetic energy.

Therefore, the total energy for the electron at the atom for the classical approach is: $E = -\left(\frac{ke^2}{2R}\right)$

Then, it is possible to obtain all the quantization formula as it was done before:

If the radiation is emitted in the transition from the initial state i to the final state f (for example from n=2 to n=1), the difference energy of those levels is as follows: $\Delta E=E_i-E_f$ $E_i>E_f$ (E_i is less negative than E_f)

$\Delta E=-\left(\frac{ke^2}{2}\left(\frac{1}{R_i}-\frac{1}{R_f}\right)\right)$ $\Delta E=\left(\frac{ke^2}{2}\left(\frac{1}{R}-\frac{1}{R_o}\right)\right)$ $R_i=R_o$ $R_f=R$

The formula of Balmer Serie is as follows:

$\frac{1}{\lambda}=R_y\left(\frac{1}{n'^2}-\frac{1}{n^2}\right)$ $n>n'$ $R_y=1.099731*10^7\ m^{-1}$: $R_y=\frac{me^4}{8\varepsilon_o{}^2ch^3}$ Rydberg Constant

$\frac{f}{c}=R_y\left(\frac{1}{n'^2}-\frac{1}{n^2}\right)$ $f=cR_y\left(\frac{1}{n'^2}-\frac{1}{n^2}\right)$ $f=\frac{me^4}{8\varepsilon_o{}^2h^3}\left(\frac{1}{n'^2}-\frac{1}{n^2}\right)$ $\lambda=c/f$

This formula was obtained from the quantization of the radius by using De Broglie wave-particle duality ($\lambda=h/mv$) and the adjustment of the wave of the electron at the orbit of the atom ($2\pi r=\lambda$) and the formula $\Delta E=hf$ for the energy emission of the electron at the atom.

Because $\Delta E=hf$, it is possible to obtain the formula of energy:

$\Delta E=hf=\frac{mhe^4}{8\varepsilon_o{}^2h^3}\left(\frac{1}{n'^2}-\frac{1}{n^2}\right)$ $\Delta E=\frac{me^4}{8\varepsilon_o{}^2h^2}\left(\frac{1}{n'^2}-\frac{1}{n^2}\right)$

This formula can be compared with the mass formula development for the classic approach: $\Delta E=\left(\frac{ke^2}{2}\left(\frac{1}{R}-\frac{1}{R_o}\right)\right)$

$$\frac{me^4}{8\varepsilon_o{}^2h^2}\left(\frac{1}{n'^2}-\frac{1}{n^2}\right)=\left(\frac{ke^2}{2}\left(\frac{1}{R}-\frac{1}{R_o}\right)\right)$$

n: main quantum number of the initial state with radius R_o

n´: main quantum number of the final state with radius R

$$\frac{e^2}{4\pi\varepsilon_o(2)\left(\frac{\varepsilon_o h^2}{\pi me^2}\right)}\left(\frac{1}{n'^2}-\frac{1}{n^2}\right)=\left(\frac{ke^2}{2}\left(\frac{1}{R}-\frac{1}{R_o}\right)\right) \qquad k=1/(4\pi\varepsilon_o)$$

$$\frac{ke^2}{2}\left(\frac{1}{\left(\frac{\varepsilon_o h^2}{\pi me^2}\right)n'^2}-\frac{1}{\left(\frac{\varepsilon_o h^2}{\pi me^2}\right)n^2}\right)=\left(\frac{ke^2}{2}\left(\frac{1}{R}-\frac{1}{R_o}\right)\right)$$

It is concluded that the radius must be proportional to the number n^2.

$R=a_o\,n'^2$ $R_o=a_o\,n^2$

$R=a_o n^2$ $a_o=\frac{\varepsilon_o h^2}{\pi me^2}$ $a_o=0{,}529*10^{-10}$ m=0,529 A Bohr radius

Besides, It is known that the radius is proportional to the main quantum number and the Bohr radius from the quantization of the radius by using the De Broglie wave-particle duality and the adjustment of the wave of the electron at the orbit of the atom: $R=\frac{\varepsilon_o h^2}{\pi me^2}n^2$. The Constant of Rydberg is obtained as follows:

$f=cR_y\left(\frac{1}{n'^2}-\frac{1}{n^2}\right)$ $\Delta E=hf=chR_y\left(\frac{1}{n'^2}-\frac{1}{n^2}\right)$ $\Delta E=\left(\frac{ke^2}{2a_o}\left(\frac{1}{n'^2}-\frac{1}{n^2}\right)\right)$

$\mathrm{ch}R_y\left(\frac{1}{n'^2}-\frac{1}{n^2}\right)=\left(\frac{ke^2}{2a_o}\left(\frac{1}{n'^2}-\frac{1}{n^2}\right)\right)$ $\mathrm{ch}R_y=\frac{ke^2}{2a_o}$

$R_y=\frac{ke^2}{2cha_o}$ h: Planck Constant $a_o=\frac{\varepsilon_o h^2}{\pi me^2}$ $k=1/(4\pi\varepsilon_o)$ c: light velocity

$R_y=\frac{me^4}{8\varepsilon_o{}^2ch^3}$ $R_y=1.0997313414*10^7\ m^{-1}$ Rydberg Constant

Besides, if we suppose that the Rydberg Constant is known from the experimental result of Balmer, then the Planck Constant is possible to obtain:

$\mathrm{ch}R_y=\frac{ke^2}{2a_o}$ $h=\frac{ke^2}{2a_o cR}$ $a_o=\frac{\varepsilon_o h^2}{\pi me^2}$

$k=1/(4\pi\varepsilon_o)=9*10^9\ N\ m^2/C^2$ ε_o: vacuum permittivity=$8,85*10^{-12}$ Farad/m

a_o: Bohr radius=$5,3*10^{-11}$ m c: light velocity=$3*10^8$ m/s

e: electron charge=$1,6*10^{-19}$ Coulomb (C)

$R_y=1.097313414*10^7\ m^{-1}$: Rydberg Constant

By using those values, it is possible to obtain the Planck Constant: $h=6,63*10^{-34}$ J-s **-----** Planck Constant

3.3. Ionization emission energy of the electrons at the Hydrogen Atom and the bound of Diatomic Molecules

In order to test the mass development formula for the ionization emission energy of the electron for the Hydrogen atom, some calculation by using Quantum Mechanics are done. After, the mass results for both methods are compared. The formulas for velocity, radius and energy for an electron at the quantized atom [24], [27], [28], [29], [30] are as follows:

$v=\frac{Ze^2}{2\varepsilon_o nh}$ $r=\frac{n^2h^2\varepsilon_o}{\pi m_o Ze^2}$ $E=\frac{-z^2e^4m_o}{8\varepsilon_o{}^2h^2n^2}$

Z: atomic number of the atom e: charge of the electron

ε_o: vacuum permittivity h: Planck constant

n: main quantum number, electron energy level, orbit of the electron

m_o: rest mass of the electron

The physical constants are given in the next table:

Table 15. Physical constants

h	$6,63*10^{-34}$ J-s
ε_o	$8,85*10^{-12}$ Farad/m
π	3,1416
m_o	9,11*10-31 kg
e	$1,6*10^{-19}$ C
r_o	0,528 $\dot{A}$

First energy level Hydrogen atom: -13,6 eV n=1

Second energy level Hydrogen atom: -3,4 eV n=2

Therefore, if the electron jumps from the first level to the second level, it must gain an energy of 10,2 eV (energy difference of the two levels). If the electron jumps from the second level to the first level, it must lose and energy of 10,2 eV. If the electron jumps from the second level to the first level, the mass of the electron must lose this equivalent mass-energy. And the lost mass of the electron (which is equivalent to the mass-energy of the electromagnetic radiation emitted) occurs during the transition from the second level to the first level converted as kinetic energy. In mathematical formulation, it is as follows: $(m_o-m)c^2=hf=K$

E=hf: energy of the photon emitted (electromagnetic radiation).

It is in coincidence with the development formula for the energy emission of the electron at the atom. Thus, we are going to proceed to test the mass formula: if the electron at the first level (n=1) leaves from the atom, then the ionizing energy is equal to -13,6 eV. It corresponds to the energy emission of the electron. Therefore, the mass of the electron after losing this mass-energy emission is: $mc^2=m_oc^2-hf$

$mc^2=511875-13,6$

$=511861,4$ eV

For other hand, the mass electron calculation with the mass development formula is as follows:

$$m = m_o e^{-(\frac{v^2}{2c^2})}$$

The velocity is given by this formula:

$$v = \frac{Ze^2}{2\varepsilon_o nh}$$

It is interesting to mention that this formula doesn't include the mass of the particle. So, the orbits have specific values for the particle independent of the mass of it. By replacing the values for Z (Z=1), e, ε_o, n (n=1), h, it is obtained: v=2181489,72 m/s. By replacing this value at the mass formula, it is achieved: $mc^2=511861,4$ eV

It is the same value that the last calculation by using quantum mechanics. It is possible to do the same for the second level of the Hydrogen atom. The ionizing energy for the electron at the second level is: -3,4 eV.

If the electron at the second level (n=2) leaves from the atom, the ionizing energy is equal to -3,4 eV. It corresponds to the energy emission of the electron. Therefore, the mass of the electron after losing this mass-energy emission is: $mc^2=m_oc^2-hf$

$mc^2=511875-3,4=511871,6$ eV

For other hand, the mass electron calculation with the mass development formula is as follows:

$$m = m_o e^{-(\frac{v^2}{2c^2})}$$

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The velocity is given by this formula:

$$v = \frac{Ze^2}{2\varepsilon_o nh}$$

By replacing the values for Z (Z=1), e, ε_o, n (n=2), h, it is obtained:

v=1090744,859 m/s

By replacing this value at the mass formula, it is achieved:

mc^2=511871,6 eV

It is the same value that the last calculation by using quantum mechanics. It is showed at the next table the values of the velocities, radius, energy of the ionization for the different levels of energy of the hydrogen atom. Also, it is showed the mass of the electron after the emission of the electromagnetic radiation by using quantum mechanics ($mc^2=m_oc^2-hf$) and for the formula of the variant mass for the electron at the atom after the energy emission: $m = m_o e^{-(\frac{v^2}{2c^2})}$. It is possible to confirm the accuracy of the formula demonstrated theoretically.

Besides, the table showed that when the velocity decreases (at the different levels of energy of the Hydrogen atom) the mass increases. Also, levels which are closest to the nucleus have higher velocities than the farthest.

Table 16. Values of the velocities, radius, energy of the ionization for the different levels of energy of the hydrogen atom. Also, it is showed the mass of the electron after the emission of the electromagnetic radiation by using quantum mechanics and for the development formula of the variant mass

n	v	r	E=hf Ecinetica (ionization energy)	$mc^2=m_oc^2-hf$	$m = m_o e^{-(\frac{v^2}{2c^2})}$
1	2181489,72	5,3096E-11	-13,54798602	511861,452	511861,4671
2	1090744,86	2,1238E-10	-3,386996504	511871,613	511871,6167
3	727163,24	4,7786E-10	-1,50533178	511873,4947	511873,4963
4	545372,43	8,4954E-10	-0,846749126	511874,1533	511874,1542
5	436297,944	1,3274E-09	-0,541919441	511874,4581	511874,4587
6	363581,62	1,9115E-09	-0,376332945	511874,6237	511874,6241
7	311641,388	2,6017E-09	-0,276489511	511874,7235	511874,7238

Bound of Diatomic Molecules

It is possible to do the analysis for the center of mass (CM) as follows:

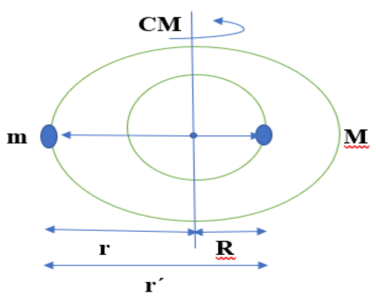

Fig.52. Center of mass for diatomic molecules

mr=MR R=(m/M)r r´=r+R r´= r+(m/M)r

$r' = \frac{M+m}{M} r \quad r = \frac{M}{M+m} r'$

$m\frac{v^2}{r} = \frac{kZe^2}{r'^2}$

$v^2 = r\frac{kZe^2}{mr'^2} = \frac{M}{M+m} r' \frac{kZe^2}{mr'^2} = \frac{kZe^2}{2m_o r'}$ M=m=m_o

$v = \sqrt{k\frac{Ze^2}{2m_o r'}}$

Firstly, we consider the Hydrogen molecule H_2. The two electrons can be shared if the spins are in opposite direction. The molecule of H_2 is more stable than the molecule of ionized hydrogen H_2^+.

The mass of H_2 is: m_o=1,67353 $*10^{-27}$ kg.

The nuclear separation is: r=0.74 A (1 A=10^{-10} m).

$v = \sqrt{k\frac{Ze^2}{2m_o r}}$ $k=\frac{1}{4\pi\epsilon_o}=9*10^9$ Nm^2/C^2 Z=1 e=1,6$*10^{-19}$ C

v=30499, 52739 m/s

$m = m_o e^{-(\frac{v^2}{2c^2})}$ where c is the light velocity c=3$*10^8$ m/s.

$\Delta mc^2=(m_o e^{-(\frac{v^2}{2c^2})})c^2- m_oc^2$

Δmc^2=4,8648 eV

The experimental value for the bond energy for the Hydrogen molecule H_2 is 4,72 eV.

For the ionized hydrogen H_2^+, it is obtained:

The mass of H_2^+ is approximately: m_o=1,67353 $*10^{-27}$ kg.

The nuclear separation is: r=1.06 A (1 A=10^{-10} m).

The bound energy for the H_2^+ is less intense than for H_2. Therefore, the nuclear separation is higher for H_2^+ than for H_2.

$v = \sqrt{k\frac{Ze^2}{2m_o r}}$ $k=\frac{1}{4\pi\epsilon_o}=9*10^9$ Nm^2/C^2 Z=1 e=1,6$*10^{-19}$ C

v=23851,53228 m/s

$m = m_o e^{-(\frac{v^2}{2c^2})}$ where c is the light velocity c=3$*10^8$ m/s

$\Delta mc^2=(m_o e^{-(\frac{v^2}{2c^2})})c^2- m_oc^2$

Δmc^2=2,9752064 eV

The experimental value for the bond energy for the Ionized Hydrogen molecule H_2^+ is 2,65 eV. The bound energy for H_2 is not the double of the bound energy for H_2^+, because the repulsion between the electrons of the H_2 decrease the bound from 5.3 eV to 4.72 eV and the distance is 0,74 A instead of 0.53 A (which is the nuclear separation of H_2^+ divided by 2: 1.06 / 2 A).

For the O_2, it is obtained:

The mass of O_2 is: $m_o=2,77 * 10^{-26}$ kg.

The nuclear separation is: r=1.21 A (1 A=10^{-10} m).

$v = \sqrt{k\frac{Ze^2}{2m_o r}}$ $k=\frac{1}{4\pi\epsilon_o}=9*10^9$ Nm^2/C^2 Z=1 $e=1,6*10^{-19}$ C

v=5862,6459,81 m/s

$m = m_o e^{-(\frac{v^2}{2c^2})}$ where c is the light velocity $c=3*10^8$ m/s

$\Delta mc^2=(m_o e^{-(\frac{v^2}{2c^2})})c^2 - m_o c^2$

Δmc^2=2,9752083 eV

It is possible to calculate the rotation frequency w for the O_2:

$L=Iw\approx\frac{h}{2\pi}$

$w\approx\frac{h}{2\pi I}$

$I=2m_o(r/2)^2=2,0277*10^{-46}$ kg m^2

By replacing the value of the Planck constant h, it is obtained:

$w=5,20*10^{11}$ Rad/s

It is in accordance with the experimental measured for the rotation frequency. The rotation frequency is lower than the vibration frequency which is in the order of 10^{13} Hz.

3.4. Conclusions

Planck's great contribution (1901) consisted in proposing that the experimental results of the blackbody radiation could be obtained if the average energy was treated as a discrete variable instead of the continuous variable of classical physics [24], [27], [31], [32], [33], [34]. The quantization of the energy of the electron oscillators of the blackbody cavity was a great advance for the atom research.

Then Rutherford proposed a planetary system for the explanation of the experiment of Geiger and Marsden. This experiment only can be explained if the nucleus is constituted by a nucleus of positive charge with the electrons with negative charge moving around it at a large distance or radius respect to the nucleus [22], [23], [26]. But, the electrons will radiate electromagnetic energy in a continue form. As consequence of it, the electrical force will put

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the electrons towards the core of the nucleus [22]. Besides, it will result in a continuous spectrum of energy emission of the electron and in instability of the atom (atom collapse) and the matter in general. But, it doesn´t occur in the reality: there is a discrete spectrum of energy emission of the electron at the atom and there is stability at the atom. Then, it was necessary to obtain other model to explain this fact.

Bohr proposed a model with some postulates to solve the instability of the atom. Bohr postulated the quantization of the energy transition for the electrons at the atom and the quantization of the angular momentum. Bohr could explain the atom stability (the no radiation for the electrons at the atom) with those postulates and obtain a formula for the quantization of the energy, velocity, radius, angular momentum, frequency and wavelength of the radiation emitted or absorbed.

Later, the modern quantum physics could explain the postulates of Bohr and obtain the quantization formula for the energy and angular momentum at the stationary levels by applying the Schröndiger Theory (wave probabilistic theory) and Heisenberg Theory (matrix theory) [22], [23]. The duality wave-particle of De Broglie and the Heisenberg Uncertainty Principle were support for the development for the modern quantum physics. The stationary states or levels correspond to those functions which satisfy the Schröndiger Equation [22], [23]. The electron in an atom no excited (electron in a stationary level) is at rest. Thus, it cannot radiate energy because it corresponds to a stationary level of the atom [22], [23].

At this research, it is to demonstrate the discovery formula which describes exactly the variant mass of a charged particle as the electron at the atom which emits electromagnetic energy from one stationary level to other: Variant mass of the electron at the Atom. The formula is in agreement at the classic limit for the bound energy for the particle orbiting the nucleus at the classic limit. The results of the formula are compared with the ionization energy emission for the electron at the atom and the bound energy for the diatomic molecules. The results of the theoretical formula are in agreement with the experimental results with high accuracy.

4. The Variant Mass for an accelerated charged particle

4.1. Introduction

Firstly, it is demonstrated by theory, calculations and results the discovered formula which describe the mass of an accelerated charged particle emitting electromagnetic radiation. Finally, thought experiments, experimental tests and calculations are presented which confirm the formula theoretically demonstrated.

4.2. Formula Development: Variant Mass for an accelerated charged particle

An accelerated charged particle emits electromagnetic radiation and then, the mass decreases [35],[36], [37]. This electromagnetic radiation emitted is the Maxwell radiation for an accelerated charged particle. It is applied the energy conservation for the development of the formula for the mass of the particle.

The change of the kinetic energy of the particle dK is transformed in an decreasing of the relativistic mass-energy $dE=c^2dm$ (by using the famous formula for the energy-mass for every particle: $E=mc^2$). Thus, it is obtained:

$$-c^2dm = dK$$

$dK = dW = Fds = \left(\frac{dp}{dt}\right) ds = dp\left(\frac{ds}{dt}\right) = d(mv)v = v^2dm + mvdv$

-$c^2dm = v^2dm + mvdv$

$-c^2dm = v^2dm + mvdv$

$(-c^2 - v^2)dm = mvdv$

$\frac{dm}{m} = \frac{v}{(-c^2-v^2)}dv$

By doing the respective integration, it is obtained:

$\int_{m_o}^{m}\frac{dm}{m} = \int_0^v -\frac{2v}{2(c^2+v^2)}dv$

By doing a variable change: $u=c^2+v^2$, $du=2vdv$,it is achieved:

$\int_{m_o}^{m}\frac{dm}{m} = \int_{c^2}^{c^2+v^2} -\frac{du}{2u}$

$\ln(m)-\ln(m_o)=(-1/2)[\ln(c^2+v^2)-\ln(c^2)]$

$\ln(m/m_o)=\ln[(c^2+v^2)/ c^2]^{-1/2}$

$m/m_o=[(c^2+v^2)/ c^2]^{-1/2}$

$m = \frac{m_o}{\sqrt{(1+\frac{v^2}{c^2})}}$

which is the formula for the mass of an accelerated charged particle which emits electromagnetic radiation.

The kinetic formula is obtained as follows:

-$c^2dm = dK$ and by doing the integration:

$-mc^2+m_oc^2=K$, then $mc^2= m_oc^2-K$ $E= m_oc^2-K$

which is the formula for the kinetic energy: $K=m_oc^2- mc^2$

$K = m_oc^2(1 - \frac{1}{\sqrt{(1+\frac{v^2}{c^2})}})$ $K = m_oc^2(1-(1+\frac{v^2}{c^2})^{-1/2})$

For very low velocities, it is obtained:

$(1+\frac{v^2}{c^2})^{-1/2} = 1 - \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4}+..$

K=$m_oc^2(1 - 1 + \frac{1}{2}\frac{v^2}{c^2} - \frac{3}{8}\frac{v^4}{c^4} + \cdots)$

$\frac{3}{8}\frac{v^4}{c^4}$ and other terms are neglected for very low velocities.

$K = \frac{1}{2}m_ov^2$ (for very low velocities as the classic formula).

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For higher velocities (when v is approaching to ∞), the kinetic energy has a maximum value of m_oc^2, which correspond to the rest mass of the particle. It is completely emitted as electromagnetic radiation at higher velocities.

$K=m_oc^2- mc^2$

$K\approx m_oc^2$ $m\approx 0$ (when $v\approx\infty$), almost all energy is emitted as electromagnetic radiation.

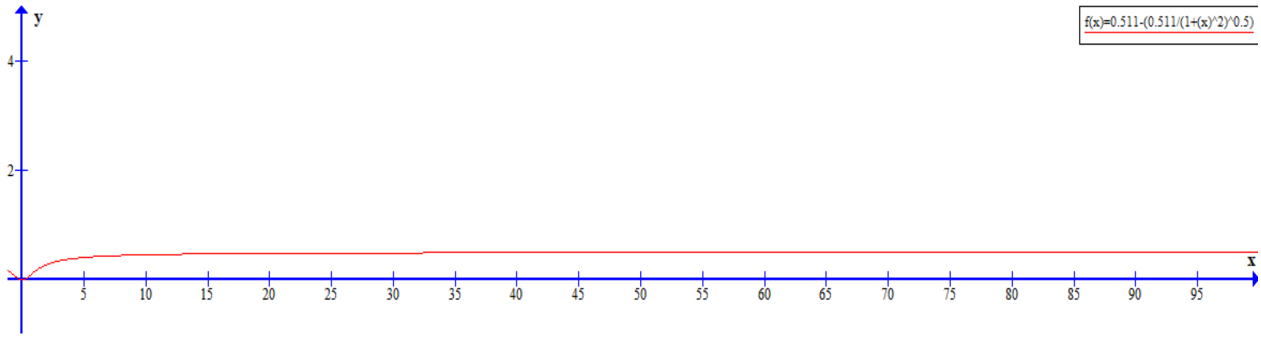

Fig.53. Graph of the kinetic energy (y axis) (MeV) versus the rate v/c (x axis) for an accelerated charged particle

At this formula, there is not limit of velocity for the accelerated charged particle. It is because the particle is losing mass (due to the emission of electromagnetic radiation) during the movement and therefore, it is possible to get each time more velocity without limit. The kinetic energy has a maximum value of m_oc^2 (0,511 MeV/c^2) when v is approaching to ∞.

With respect to the mass formula, it is possible to obtain the next conclusion:

$$m=\frac{m_o}{\sqrt{1+\frac{v^2}{c^2}}}$$

Where, m_o is the mass at rest of the particle.

At this formula, when the velocity v is increasing, the mass m is decreasing too until v is very high (∞) and the value of mass is very low (approximately to 0). At this point, almost all the mass is emitted (transformed) as electromagnetic energy. Thus, the velocity v can get values higher than the light velocity because of the decreasing mass. The lost mass is equivalent to the mass-energy of the electromagnetic radiation emitted.

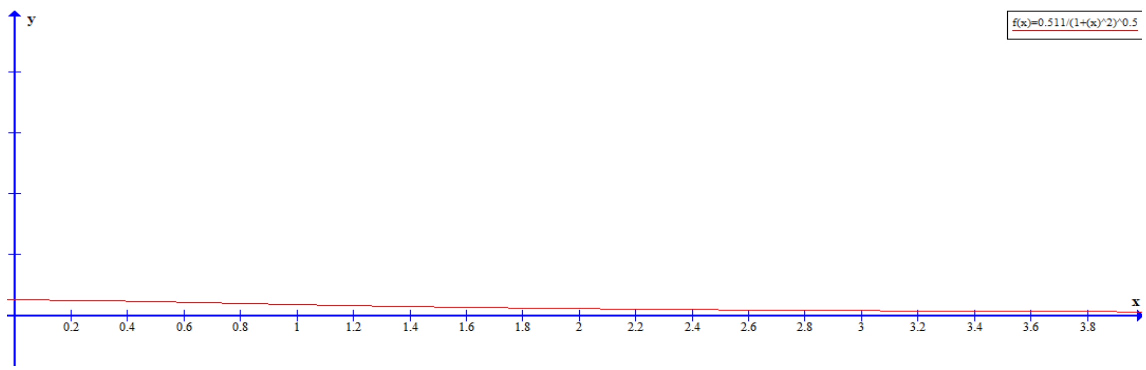

Fig.54. Graph of the mass (y axis) (MeV/c^2) versus the rate v/c (x axis) for an accelerated charged particle. The initial value of the mass is 0,511 MeV/c^2

4.3. Thought Experiment (Gedanken experiment)

Thought Experiment (Gedanken experiment 1)

* A rectangular tube of mass M and length L is at rest in a system S. A pulse of electromagnetic radiation of energy E is emitted from one extreme of the tube, and then it is absorbed at the other extreme [38]. In this thought experiment, it is demonstrated as follows that the inertia associated with this radiation is m = E / c^2.

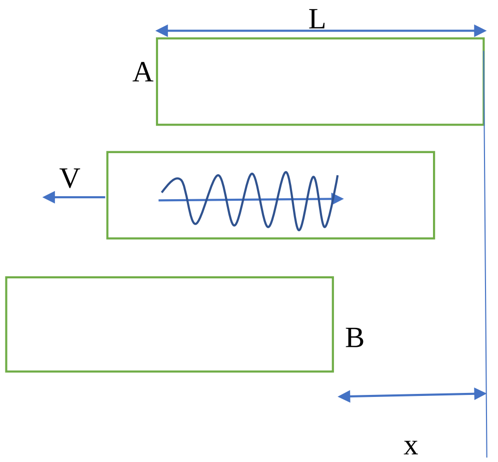

Fig.55. Emission of the electromagnetic radiation pulse (in A) within of the tube of length L. The tube recoils due to the emission of the pulse (momentum conservation). The pulse reaches B and then leaves the tube

Firstly, it is used momentum conservation and then energy conservation as a second form to demonstrate that the inertia associated with this radiation is m = E / c^2.

Momentum Conservation

The electromagnetic radiation pulse is emitted to the right. From Maxwell's theory of electromagnetism it is known that an impulse p = E / c is associated with this radiation. In order to preserve the momentum (amount of movement), the tube moves back to the left with velocity v. If m represents the mass associated with the radiated energy, then the mass of the receding tube is M-m and the conservation of momentum in S requires:

$$(M-m)v = \frac{E}{c} \quad v = \frac{E}{(M-m)c}$$

(p=E/c: impulse of the photon or electromagnetic radiation pulse)

The flight time of the electromagnetic radiation pulse whose velocity is c, is t=(L-x)/c (where x is the displacement of the tube to the left due to the inertia: recoil of the tube). This time is equal to the recoil time of the tube, t=x/v. Combining these times, it is obtained:

$$\frac{(L-x)}{c} = \frac{x}{v}$$

$$\frac{v}{c} = \frac{x}{(L-x)}$$

By replacing the value of v, it is obtained:

$$\frac{E}{(M-m)c^2} = \frac{x}{(L-x)}$$

Since all forces are internal, the center of mass of the system does not change during the emission and absorption processes. If m is the effective mass emitted by radiation, the center of mass will not change if:

Mx=mL

Demonstration:

M(L)=(M-m)(L+x)+mx

ML=ML+Mx-mL-mx+mx

Mx=mL

$$m = \frac{Mx}{L} \quad x = \frac{mL}{M}$$

$$\frac{E}{(M-m)c^2} = \frac{x}{(L-x)}$$

By solving for x, it is achieved:

$$x = \frac{EL}{(M-m)c^2+E}$$

And also: $x = \frac{mL}{M}$

By equating both equations and solving for m, it is obtained:

$$\frac{EL}{(M-m)c^2+E} = \frac{mL}{M}$$

$$m^2 - \left(\frac{E}{c^2} + M\right)m + \frac{ME}{c^2} = 0$$

There are two mathematical solutions:

$m=E/c^2$ and m=M, but the second solution is not valid because m≠M.

Thus, it has been shown that the electromagnetic radiation pulse has an associated mass $m = E/c^2$ (which previously in the case of the tube belonged to the tube (it is the lost mass in the tube) and was emitted as electromagnetic radiation).

It is possible to get a very important conclusion [38], [39], [40]:

How does the message of moving to the left arrives to the right extreme of the tube after the left extreme of the tube goes back?

This message of moving to the left for the right extreme of the tube must travel much faster than the light velocity to win the pulse emitted. If the tube were to move like a rigid body the message would travel at infinite velocity. The entire tube does not go back rigidly. So, this appears that the message of mass lost in the tube m (mass-energy emitted as a pulse of electromagnetic energy travelling at the light velocity) is transmitted along the tube at infinite

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velocity. This is in accordance with the mass formula for a charged particle that emits electromagnetic radiation because the particle loses approximately all its mass at very high speeds and when v approaches infinity the mass approaches zero. In other words, this lost mass in the tube is related to the infinite transmission of the message along the tube (the lost mass is transmitted along the tube at infinite velocity).

Energy Conservation

$mc^2=pc$ The energy of the photon emitted is E=pc and m is the lost mass of the tube and emitted as electromagnetic radiation.

M-m is the mass of the tube after the emission of the electromagnetic pulse.

E=hf (energy of the photon)

E=hc/λ (c= λf for the light velocity) & p=h/ λ (for a photon)

Therefore, E=pc for a photon.

Also, it is possible to obtain the same from the Dirac equation: $E^2 = m^2c^4 + p^2c^2$

As the mass of the photon is m=0, then, it is obtained E=pc

Thus, the energy of the lost mass of the tube is equal to the energy of the photon emitted: $mc^2=pc$

Therefore: p=mc and because the energy of the photon is E=pc, it is obtained:

$E=mc^2$ and the associated mass of the photon emitted is $m=E/c^2$, where E is the energy of the photon and m is the lost mass on the tube and also the associated mass-energy of the photon emitted.

Thought Experiment (Gedanken experiment 2)

*Consider a body B at rest in system S. Two pulses of radiation, each one with E/2 energy, are emitted from B, one pulse moving in the positive x direction and the other in the negative x direction.

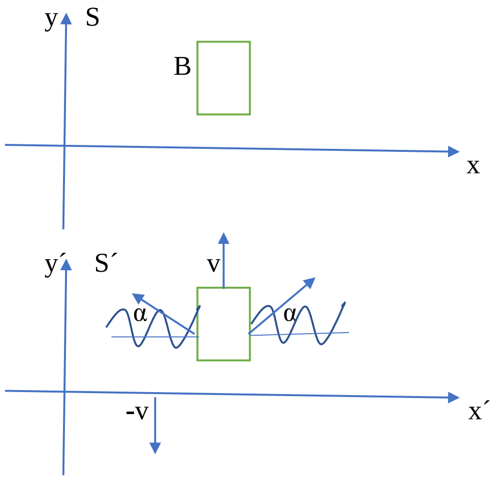

Fig.56. The body B is at rest in S. Then two pulses of radiation are emitted in the directions shown by the arrows. In S´ that moves with respect to S with velocity v in the negative direction of y, the body B moves with velocity v along + y´ and the pulses are directed making an angle α with the x-axis

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These pulses are emitted by B, whose energy then decreases by a quantity E and E/2 is the energy of each pulse emitted. For reasons of symmetry, B must remain at rest in S. Now, let us consider the same process with respect to S´, which moves with a constant velocity v with respect to S in the negative direction of y. In this case, B moves with velocity v in the positive direction of y '. The radiation pulses are partly directed upward, forming an angle α with the x' axis. The velocity of B remains constant in S´ (after the emission of the radiation) since B remains at rest in S during this process. Now, let's apply the law of momentum conservation in the process in S´. The momentum of B in S´ before the emission is Mv (from classic mechanics) along the positive direction of y´. In other words, before the emission takes place, the y´ component of the impulse of B is: Mv

After the emission, the body B has mass M´ and the momentum is M´v along the positive direction y´. Each radiation pulse emitted has an energy of E / 2 and a momentum p=E/2c (p = E/c from classic electromagnetism) with the components of momentum being equal to (E / 2c) sin α .

Therefore, by applying momentum conservation after the emission, it is obtained: M´v+2(E/2c)sin α

By equating the components y´ of the impulse before the absorption with those of after absorption, it is achieved:

Mv= M´v+2(E/2c)sin α

sinα≈ α =v/c (v<<c) (classic absorption)

Mv= M´v+2(E/2c)(v/c)

$M=M'+(E/c^2)$

$M-M'=E/c^2$

Therefore, the decrease in mass ΔM=M-M´ of the body B is linked to the energy E of the two photons emitted (summed energy) (this energy is also the lost energy of the body B). Therefore:

$E=(M-M')c^2$

$E=\Delta Mc^2$

$\Delta M=E/c^2$ and the associated mass of the photon emitted is $\Delta M=E/c^2$, where E is the energy of the photon emitted and ΔM is also the lost mass on the body B.

4.4. Experimental Test

Every experiment where it is possible to measure the kinetic energy for an accelerated charged particle emitting electromagnetic radiation can be a test for the formula:

$$K = m_o c^2 (1 - \frac{1}{\sqrt{(1+\frac{v^2}{c^2})}})$$

In fact, a signal or prove of this formula is the maximum kinetic energy of the accelerated charged particle that can be measured. It is $m_o c^2$ if there is not additional energy given to the particle at the beginning.

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This energy can be measured at the PMT (photomultiplier) as the maximum kinetic energy for an electron for example.

Also, it is an intrinsic characteristics of the charged particle and independent of the experimental configuration or material used as detector at the experiment for example.

This maximum kinetic energy for the electron appears in the experiment of muons and neutral kaons from ultra dense hydrogen by lepton pair production [39].

At this experiment, we have the next decays:

$\mu \pm \rightarrow N(e^+ + e^-)$

$\mu \pm \rightarrow \mu\pm + N(e^+ + e^-) + KE$ KE: 10-100 MeV dissipation

$\mu \pm \rightarrow e\pm +$ neutrinos $+ N(e^+ + e^-) + KE$ KE<105 MeV dissipation

The energy for the pair-production is derived from the kinetic energy of the muons of 10-100 MeV and from the decay of the muons at 105 MeV. As the dissipation energy is around 100 MeV and each pair production process dissipates around 2.04 eV, then we have around 50 lepton pair per muon [39]. The kinetic energy given to the electron-positron pairs increases until that energy is equal to 2x511 keV when a new pair can be produced and energy above this limit gives a new lepton pair[39]. Therefore, at the beginning the electron has a energy equals to the rest mass of the particle.

The accelerated electron has many interactions with the atoms and nuclei of the material used as detector but nevertheless the maximum kinetic energy measured at the PMT is approximately 511 keV [39]. It is the radiated energy for the electron during the path. Also, the same electron energy distribution is found for all materials used as converters [39]. Thus, it depends of an electron property and no of the converter material used [39]. It is in accordance with the kinetic formula for the electron which we have demonstrated before.

It is a intrinsic property of the electron: the way that the electron radiates the energy, specifically the electrodynamic mass. Just the high energy cut-off is close to the rest mass of the electron 511 keV, which is a confirmation of the nature of electron when it shares the electromagnetic radiation. Therefore, the kinetic energy of leptons entering and measured at the PMT is approximately 511 keV (zero-signal cut-off above 511 keV [39]) which is in agreement with the formula demonstrated.

4.5. Hydrogen atom: Emission of electromagnetic radiation for the electron at the atom

In order to test the mass formula for an accelerated charged electron for the Hydrogen atom, some calculation by using Quantum Mechanics are done. After, the mass results for both methods are compared. The formulas for velocity, radius and energy for an electron at the quantized atom [40], [41], [42], [43], [44] are as follows:

$$v = \frac{Ze^2}{2\varepsilon_o nh}$$

$$r = \frac{n^2h^2\varepsilon_o}{\pi m_o Ze^2}$$

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$$E = \frac{-z^2 e^4 m_o}{8 \varepsilon_o{}^2 h^2 n^2}$$

Z: atomic number of the atom

e: charge of the electron

ε_o: vacuum permittivity

n: main quantum number, electron energy level, orbit of the electron

h: Planck constant

m_o: rest mass of the electron

Values of constants:

Z=1

$h=6,63*10^{-34}$ J-s

$\varepsilon_o=8,85*10^{-12}$ Farad/m

$\pi=3,1416$

$m_o=9,11*10^{-31}$ kg

$e=1,6*10^{-19}$ C

$r_o=0,528$ $\dot{A}$ Bohr radius, radius for the first level of the Hydrogen atom

1 $\dot{A}$= 10^{-10}m

Table 17. Table of physical constants

h	$6,63*10^{-34}$ J-s
ε_o	$8,85*10^{-12}$ Farad/m
π	3,1416
m_o	9,11*10-31 kg
e	$1,6*10^{-19}$ C
r_o	0,528 $\dot{A}$

First energy level Hydrogen atom: -13,6 eV n=1

Second energy level Hydrogen atom: -3,4 eV n=2

Therefore, if the electron jumps from the first level to the second level, it must gain an energy of 10,2 eV (energy difference of the two levels).

If the electron jumps from the second level to the first level, it must lose and energy of 10,2 eV.

If the electron jumps from the second level to the first level, the mass of the electron must lose this equivalent mass-energy. And the lost mass of the electron (which is equivalent to the mass-energy of the electromagnetic radiation emitted) occurs during the transition from the second level to the first level converted as kinetic energy.

In mathematical formulation, it is as follows:

$(m_o\text{-}m)c^2=hf=K$

E=hf: energy of the photon emitted (electromagnetic radiation).

It is in coincidence with the formula for the kinetic energy for an Accelerated Charged Particle.

Thus, we are going to proceed to test the mass formula:

If the electron at the first level (n=1) leaves from the atom, then the ionizing energy is equal to -13,6 eV. It corresponds to the energy emission of the electron.

Therefore, the mass of the electron after losing this mass-energy emission is:

$mc^2=m_oc^2\text{-}hf$

$mc^2=511875\text{-}13,6$

=511861,4 eV

For other hand, the mass electron calculation with the mass formula for an accelerated charged particle is as follows:

$$m = \frac{m_o}{\sqrt{1+\frac{v^2}{c^2}}}$$

The velocity is given by this formula:

$$v = \frac{Ze^2}{2\varepsilon_o nh}$$

It is interesting to mention that this formula doesn't include the mass of the particle. So, the orbits have specific values for the particle independent of the mass of it.

By replacing the values for Z (Z=1), e, ε_o, n (n=1), h, it is obtained:

v=2181489,72 m/s

By replacing this value at the mass formula, it is achieved:

$$mc^2 = \frac{m_oc^2}{\sqrt{1+\frac{v^2}{c^2}}}$$

mc^2=511861,4 eV It is the same value that the last calculation by using quantum mechanics.

It is possible to do the same for the second level of the Hydrogen atom. The ionizing energy for the electron at the second level is: -3,4 eV.

If the electron at the second level (n=2) leaves from the atom, the ionizing energy is equal to -3,4 eV. It corresponds to the energy emission of the electron.

Therefore, the mass of the electron after losing this mass-energy emission is:

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$mc^2=m_oc^2-hf$

$mc^2=511875-3,4$

$=511871,6$ eV

For other hand, the mass electron calculation with the mass formula for an accelerated charged particle is as follows:

$$m = \frac{m_o}{\sqrt{1+\frac{v^2}{c^2}}}$$

The velocity is given by this formula:

$$v = \frac{Ze^2}{2\varepsilon_o nh}$$

By replacing the values for Z (Z=1), e, ε_o, n (n=2), h, it is obtained:

v=1090744,859 m/s

By replacing this value at the mass formula, it is achieved:

$$mc^2 = \frac{m_oc^2}{\sqrt{1+\frac{v^2}{c^2}}}$$

$mc^2=511871,6$ eV It is the same value that the last calculation by using quantum mechanics. It is showed at the next table the values of the velocities, radius, energy of the ionization for the different levels of energy of the hydrogen atom. Also, it is showed the mass of the electron after the emission of the electromagnetic radiation by using quantum mechanics ($mc^2=m_oc^2-hf$) and for the formula of the variant mass for an accelerated charged particle: $mc^2 = \frac{m_oc^2}{\sqrt{1+\frac{v^2}{c^2}}}$. It is possible to confirm the accuracy of the formula demonstrated theoretically. Besides, the table showed that when the velocity decreases (at the different levels of energy of the Hydrogen atom) the mass increases. Also, levels which are closest to the nucleus have higher velocities than the farthest.

Table 18. Values of the velocities, radius, energy of the ionization for the different levels of energy of the hydrogen atom. Also, it is showed the mass of the electron after the emission of the electromagnetic radiation by using quantum mechanics and for the formula of the variant mass for an accelerated charged particle

n	v	r	E=hf (ionizatin energy)	$mc^2=m_oc^2-hf$	$mc^2 = \frac{m_oc^2}{\sqrt{1+\frac{v^2}{c^2}}}$
1	2181489,719	5,30961E-11	-13,54798602	511861,452	511861,4674
2	1090744,859	2,12384E-10	-3,386996504	511871,613	511871,6168
3	727163,2396	4,77865E-10	-1,50533178	511873,4947	511873,4963
4	545372,4297	8,49537E-10	-0,846749126	511874,1533	511874,1542
5	436297,9438	1,3274E-09	-0,541919441	511874,4581	511874,4587
6	363581,6198	1,91146E-09	-0,376332945	511874,6237	511874,6241
7	311641,3884	2,60171E-09	-0,276489511	511874,7235	511874,7238

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4.6. Power energy emitted for an Accelerated Charged Particle

The kinetic formula for an accelerated charged particle is as follows:

$$K = m_o c^2 - mc^2$$

$$K = m_o c^2 \left(1 - \frac{1}{\sqrt{(1+\frac{v^2}{c^2})}}\right)$$

$\frac{dK}{dt} = -c^2 \frac{dm}{dt}$ (the change in the kinetic energy of the particle causes a proportional change in its mass, the minus sign is due to the mass decreases)

$$K = m_o c^2 (1 - (1 + \frac{v^2}{c^2})^{-1/2})$$

By doing the respective derivation, it is obtained:

P=dk/dt

$$\frac{dK}{dt} = \frac{m_o v}{(1+\frac{v^2}{c^2})^{3/2}} \frac{dv}{dt}$$

$P = \frac{m_o v}{(1+\frac{v^2}{c^2})^{3/2}} a$ (formula of the power energy for an accelerated charged particle).

The term in parenthesis can be expressed as follows:

$$\left(1 + \frac{v^2}{c^2}\right)^{-\frac{3}{2}} = 1 - \frac{3}{2}\frac{v^2}{c^2} + \frac{15}{8}\frac{v^4}{c^4} - \cdots$$

For v<<c, it is achieved:

$P = m_o v\, a \left(1 - \frac{3}{2}\frac{v^2}{c^2}\right)$ for very low velocities

If this formula is compared with the Larmor formula (which is used to calculate the power radiated by a non-relativistic point accelerated charge, electromagnetic theory [41], [42], [43], [44]):

$P = \frac{q^2 a^2}{6\pi\varepsilon_o c^3}$ or in CGS units: $P = \frac{2q^2 a^2}{3c^3}$

$$\frac{q^2 a^2}{6\pi\varepsilon_o c^3} = m_o v \left(1 - \frac{3}{2}\frac{v^2}{c^2}\right) a$$

$$a = \frac{6\pi\varepsilon_o c^3}{q^2} m_o v \left(1 - \frac{3}{2}\frac{v^2}{c^2}\right)$$

Where, q is the charge of the particle.

Or if it is used the equivalent formula:

$P = \frac{2}{3}\frac{m_o r_e a^2}{c}$, it is obtained:

$$\frac{2}{3}\frac{m_o r_e a^2}{c} = m_o v \left(1 - \frac{3}{2}\frac{v^2}{c^2}\right) a$$

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$a = \frac{3vc}{2r_e}\left(1 - \frac{3}{2}\frac{v^2}{c^2}\right)$ where r_e is the classic electron radius: $r_e=2{,}82*10^{-15}$ m

4.7. Conclusions

The kinetic formula represents the emission of the electromagnetic radiation for an accelerated charged particle. It is:

$$K = m_o c^2 (1 - \frac{1}{\sqrt{(1+\frac{v^2}{c^2})}})$$

$K=m_oc^2- mc^2$

K=hf (it corresponds to the ionization energy at the atom of Hydrogen or the energy of the photon emitted or the electromagnetic radiation emitted by the accelerated charged particle). For higher velocities (when v is approaching to ∞), the kinetic energy has a maximum value of m_oc^2, which correspond to the rest mass of the particle. It is completely emitted as electromagnetic radiation at higher velocities. The mass of the particle after the emission of the electromagnetic radiation (losing the respective mass-energy of the electromagnetic radiation and equal to the lost mass of the particle) is:

$m = \frac{m_o}{\sqrt{1+\frac{v^2}{c^2}}}$ (mass of the particle after the emission of the electromagnetic radiation)

$m = \frac{m_o c^2}{\sqrt{1+\frac{v^2}{c^2}}}/c^2$ (mass of the particle in units of energy: eV/c^2)

Also, it is equals to:

$mc^2=m_oc^2$- (hf) where hf is the energy of the photon emitted or emission of the energy of the electromagnetic radiation.

At the mass formula, when the velocity v is increasing, the mass m is decreasing too until v is very high (∞) and the value of mass is very low (approximately to 0). At this point, almost all the mass is emitted as electromagnetic energy. Thus, the velocity v can get values higher than the light velocity because of the decreasing mass. The lost mass is equivalent to the mass-energy of the electromagnetic radiation.

These formulas have been verified by means of theory, thought experiments, experiments and in the hydrogen atom.

The formula of the power energy for an accelerated charged particle is as follows:

$$P = \frac{m_o v}{(1+\frac{v^2}{c^2})^{3/2}} a$$

For very low velocities, it is as follows:

$$P = m_o v\, a\left(1 - \frac{3}{2}\frac{v^2}{c^2}\right)$$

The acceleration for very low velocities is as follows:

$a = \frac{6\pi\varepsilon_o c^3}{q^2} m_o v \left(1 - \frac{3}{2}\frac{v^2}{c^2}\right)$ where q is the charge of the particle

or:

$a = \frac{3vc}{2r_e}\left(1 - \frac{3}{2}\frac{v^2}{c^2}\right)$ where r_e is the classic electron radius.

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