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The Turbulent Lagrangian Dissipative Particle Velocity Statistics

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ABSTRACT

A statistical description of turbulence comprising a probability distribution for stationary flows is the basis for determining the nature of turbulent velocity fluctuations. Away from the Eulerian spatial increments of the velocity field, the Lagrangian notion of temporal velocity increments along particle trajectories appears to be the hob of turbulent velocity statistics, ostensibly in the presence of inherent intermittencies observable in particle tracking. Such intermittencies were analyzed by using the well-known velocity structure functions, which defer to the Kolmogorov similarity theory (KST). Since particle trajectories, under normal circumstances, submit to Gaussian statistics, the deviations from Kolmogorov similarity theory are considered a culprit in the event of intermittency.

Keywords: *Auto-correlation; Diffusion; Dissipation; Probability; Weiner process.*

1. INTRODUCTION

The understanding of the instantaneous motions of turbulence is problematical owing to frequent unexpected changes. The central process in turbulent flows is convection induced by the instantaneous fluid velocity. Molecular diffusion fails to afford an impressive contribution to spatial transport at a high Reynolds number, and therefore convection controls the transference of momentum, chemical species, and enthalpy. Statistical measures hold well for the description of turbulent motion.

A statistical description of turbulence comprises a probability distribution for stationary flows to determine the nature of turbulent velocity fluctuations. To study the statistical properties of turbulence Monin and Yaglom [1] enunciated the velocity structure functions (VSF) which are directly related to the probability density function (PDF) of the local dissipation rate. Therefore, an appropriate model for the PDF of the dissipation rate completely describes the statistics of turbulent velocity.

The presence of intermittent behaviour of turbulent Lagrangian velocity statistics was discovered in various investigations on particle tracking [2, 3]. The massive progress in experimental procedures [4, 5] of measuring particle trajectories has caused a speedy development in the dynamics of tracer particles in incompressible flows. Some intriguing features of Lagrangian turbulence such as the notable role of coherent structures or almost singular structures compared to the Eulerian description have been revealed by numerical simulations [6], multifractal, and PDF-modeling [7,8].

In Arn`eodo *et al.* [9] an elegant collection of eight data sets from experiments and numerical simulations on turbulent velocity statistics along particle trajectories was carried out. The multifractal theory, which was extended to the dissipative scales and the Lagrangian domain, was seen to hold well for the studies in the intermittency of velocity statistics that were investigated. Intermittency attributes the multi-time correlations to the turbulence-rousing force. In effect, it was used in modelling the trajectory of a fluid particle by a *multifractal* *random walk* [10] wherein long-time correlations and the incidence of large amplitude events at small scales dominate the motion. Based on this, the multifractal properties of the Lagrangian velocity may be explained. Lagrangian intermittency is associated with the nature, distribution, and time evolution of the dynamical structures entrenched in the flow. Such intermittencies were studied by using the velocity structure functions, which submit to the Kolmogorov similarity theory (KST). In fine, the central quantities of concern in the statistical theory of Lagrangian turbulence are the pdfs of the fluid particles [11, 12, 13, 14, 15].

2. LAGRANGIAN DEPICTION OF TURBULENCE

The notion of temporal velocity increments along particle trajectories, $\delta v_i^L(\zeta) = v_i^L(t+\zeta) - v_i^L(t)$ which is the Lagrangian analogue of the Eulerian spatial increments of the velocity field can be introduced as an extension of the K41 phenomenology, originally advanced in the Eulerian framework. Here, v^L encodes the Lagrangian velocity, measured for each fluid parcel along its trajectory. The Statistics are assumed to be only dependent on the time increment, ζ (i.e. stationary). The multi-scale temporal dynamics of Lagrangian temporal may be examined by observing the spanning statistical behaviours at different values of ζ. The multi-scale classification is largely realized by considering the dependence of the statistical moments of Lagrangian increments with the time increment, ζ . These define the Lagrangian function structures: $G_p^L(\zeta) = \langle \delta v_i^L(\zeta)^p \rangle$.

2.1. Equations of motion

Let **X**(t;**v**,**y**) be a transformation that maps the initial position **X**(t = 0; **v**, **y**) = y of a hypothetical tracer particle with initial velocity **v** onto its position **X** at later times t > 0. The equations of motion of a single particle are of the form [12]:

$$\frac{d\mathbf{X}(t;\mathbf{v},\mathbf{y})}{dt} = \mathbf{U}(t;\mathbf{v},\mathbf{y}) \tag{1}$$

$$\frac{d\mathbf{U}(t;\mathbf{v},\mathbf{y})}{dt} = \mathbf{A}(t;\mathbf{v},\mathbf{y}) \tag{2}$$

where in (1) *d/dt* is the time derivative along the particle's trajectory and the particle's acceleration. In (2) *d/dt* is the Lagrangian or material time derivative that may relate to the partial derivatives of the Eulerian fields (see [16])

$$\frac{d\mathrm{U}_i}{dt}(t) = \frac{\partial u_i}{dt}(\mathbf{X}(t),t) + u_j(\mathbf{X}(t),t)\frac{\partial u_i}{\partial x_j}(\mathbf{X}(t),t) \tag{3}$$

The acceleration, **A** in (2) can be expressed in terms of the Eulerian pressure and the velocity field estimated at the location of the particle. The particle's position and velocity in the Eulerian sense, having the velocity field ***u***(**x**, t), related to the Lagrangian sense reads:

$$\frac{d\mathrm{X}_i}{dt}(t) = U_i(t) = u_i(\mathbf{X}(t),t) \tag{4}$$

and in a similar mode, the particle's acceleration **A**(t) reads

$$A_i(t) = \frac{d\mathrm{U}_i}{dt}(t) = \frac{d^2 X_i}{dt^2}(t) \tag{5}$$

In the Lagrangian mode, Navier–Stokes equations can be written in the form:

$$\frac{dU_i}{dt}(t) = \frac{1}{\rho}\frac{\partial p}{dx_i}(\mathbf{X}(t),t) + \nu\frac{\partial^2 u_i}{\partial x_j \partial x_j}(\mathbf{X}(t),t), \quad (6)$$

$$\frac{\partial u_i}{\partial x_i} = 0. \quad (7)$$

2.2. Lagrangian pdf.

Let $\mathbf{x}_0 = \{x_{01}, x_{02}, x_{03}\}$ be Lagrangian coordinates and let t_0 be a reference time. Therefore $\mathbf{x}^L(t, \mathbf{x}_0)$ denotes the position at time t of the fluid particle that is at x_0 at time t_0 [i.e. $\mathbf{x}^L(t_0, \mathbf{x}_0) = \mathbf{x}_0$; where for each Eulerian variable $\mathbf{U}(\mathbf{x}, t)$ say, the corresponding Lagrangian variable is denoted by the superscript (or subscript) L]. From Pope [17] we get

$$\mathbf{U}^L(t,\mathbf{x}_0) = \mathbf{U}\left[(t,\mathbf{x}_0),t\right] \quad (8)$$

Consequently, for the Eulerian velocity, $\mathbf{x}^L$ is the solution to the initial value problem,

$$\left.\begin{aligned} \frac{\partial \mathbf{x}^L(t,\mathbf{x}_0)}{\partial t} &= \mathbf{U}^L(t,\mathbf{x}_0), \\ \mathbf{x}^L(t,\mathbf{x}_0) &= \mathbf{x}_0 \end{aligned}\right\}. \quad (9\text{ a, b})$$

Considering a constant (density) property flow [17, 18, 19], the Lagrangian joint pdf of the fluid particle properties at time t, conditional upon their properties at time t_0, for the event

$$\{\mathbf{U}^L(t,\mathbf{x}_0) = \mathbf{V},\ \mathbf{x}^L((t,\mathbf{x}_0)) = \mathbf{x}\} \quad (10)$$

is given by

$$g_L\left(\mathbf{V},\mathbf{x};t \middle| \mathbf{V}_0,\mathbf{x}_0\right), \quad (11)$$

subject to the condition $\mathbf{U}^L(t_0,\mathbf{x}_0) = \mathbf{V}_0$. The relation (12) is analogous to the (one-point, one-time Eulerian) joint pdf of velocity $g_u(\mathbf{V}; \mathbf{x}, t)$. In general, the joint pdf of the events

$$\{\mathbf{U}^L(t_r,\mathbf{x}_0) = \mathbf{V}_r,\ \mathbf{x}^L(t_r,\mathbf{x}_0) = \mathbf{x}_r; r = 1,2,\ldots,N\} \quad (12)$$

for N times, subject to the initial condition, $\mathbf{U}^L(t_0, \mathbf{x}_0) = \mathbf{V}_0$, is

$$g_{L,N}\left(\mathbf{V}_N,\mathbf{x}_N;t_N : \mathbf{V}_{N\text{-}1},\mathbf{x}_{N\text{-}1};t_{N-1};\ldots;\mathbf{V}_1,\mathbf{x}_1;t_1 \middle| \mathbf{V}_0,\mathbf{x}_0\right). \quad (13)$$

3. FLUID-PARTICLE VELOCITY

The analysis of one component of the fluid-particle velocity $\mathbf{U}^L(t)$ leads to a stochastic differential equation (SDE) that represents the Langevin equation, $\mathbf{U}^*(t)$. The crux of the velocity stochastic turbulence models is in the decomposition of a particle's acceleration in the form

$$\frac{d\mathbf{U}}{dt} = \xi\mathbf{A}(t) + \eta\mathbf{A}(t), \quad (14)$$

where $\xi\mathbf{A}(t)$ and $\eta\mathbf{A}(t)$ encode the slow acceleration and fast acceleration respectively (see Sawford and Guest [20]. Whereas $\xi\mathbf{A}(t)$ evolves on the same timescale τ_u as the velocity $\mathbf{U}^{\mathbf{L}}$ with τ_u being of the order of k/ε , $\eta\mathbf{A}(t)$ evolves on the timescale τ_a which is of the order of the Kolmogorov timescale τ_ω.

In a fully developed turbulence $\tau_a << \tau_u$, and therefore for the large scales,

$$\mathrm{Re} = \left(\frac{\tau_u}{\tau_a}\right)^2 . \tag{15}$$

The generalized Langevin model for a velocity stochastic turbulence, in line with (14) is of the form [21, 22, 16]

$$\frac{dU_i}{dt} = \langle a_i \rangle + \boldsymbol{T}_{ij} U_j' + C_{ij}\frac{dW_j}{dt} \tag{16}$$

In (16) above we let $\xi\mathbf{A}_i = \langle a_i \rangle + \boldsymbol{T}_{ij} U_j'$, and $\eta\mathbf{A}_i = C_{ij}\frac{dW_j}{dt}$

wherein

$$\xi\mathbf{A}_i(t) = \langle a_i \rangle(\mathbf{X}(t),t) + T_{ij}(\mathbf{X}(t),t)\left[U_j(t) - \langle u_j \rangle(\mathbf{X}(t),t)\right] . \tag{17}$$

encodes the slow acceleration; T_{ij} is the drift tensor which depends only on the location of the particle. The white noise that models the fast acceleration $\eta\mathbf{A}_i(t)$ is demonstrated by

$$\eta\mathbf{A}_i(t) = D_{ij}(\mathbf{X}(t),t)\frac{dW_j}{dt}(t) \tag{18}$$

where D is the diffusion tensor which also depends only on the location of the particle, and $W(t)$ is a vector-valued Wiener process. For an insight into the diffusion tensor, let us consider a coupled stochastic process in velocity, $\boldsymbol{u}$ and acceleration $\boldsymbol{a}$ of the form (see Zamansky [23]):

$$\begin{aligned} du_i &= a_i dt, \\ da_i &= B_i dt + D_{ij} dW_j \end{aligned} \tag{19a,b}$$

where dW_j are the increments of the jth component of the Wiener process ($\langle dW_j \rangle = 0$; $\langle dW_i dW_j \rangle = dt\delta_{ij}$); each of the vector B (the drift) and the diffusion tensor D depend on the vectors a and u. The tensor D can be decomposed into

$$D_{ij} = d_1\delta_{ij} + S_{ij} + \Omega_{ij}, \quad (d_1 \text{ not a constant}) , \tag{19c}$$

where S_{ij} encodes a zero-trace symmetric tensor and $\mathbf{\Omega}_{ij}$ is an antisymmetric tensor. For statistical isotropy consideration of the acceleration, S_{ij} must vanish but $\mathbf{\Omega}_{ij}$ need not be 0. The expressions for B_i and D_{ij} are elegantly specified in [23]. Now, consider a stationary homogeneous isotropic turbulence, with zero mean velocity, turbulence intensity (TI) u', and T the Lagrangian integral time scale. A one-component Langevin equation, in $\mathbf{U}^*$, of the fluid-particle velocity $\mathbf{U}^{\mathrm{L}}(t)$ reads

$$dU^*(t) = -U^*(t)\frac{dt}{T} + (2u'^2/T)^{1/2} dW(t) \tag{19d}$$

with the *rms* fluid-particle velocity as u'. Note that $\mathbf{U}^*(t)$ indicate the modelled particle properties for $\mathbf{U}^L(t)$.The Lagrangian velocity auto-correlation $R_L(t, s)$ given by

$$R_L(t,\zeta) = \frac{\langle U^L(t)U^L(t+\zeta)\rangle}{\left(\langle (U^L)^2(t)\rangle\langle (U^L)^2(t+\zeta)\rangle\right)^{1/2}} \cdot \quad (20)$$

However, in stationary homogeneous turbulence, the Lagrangian velocity auto-correlation, $R_L(\zeta)$ depends only on the time lag, ζ, and is an even C^2-function (see [24]). Thus, at the origin $R_L'(\zeta = 0) = 0$. The auto-correlation in **(20)** is the exponential function

$$R(\zeta) = e^{-\zeta/T} \Rightarrow R'(\zeta = 0) = -1/T \neq 0 \text{ ,} \quad (21)$$

and the time scale T satisfies

$$T = \int_0^\infty R_L(\zeta)d\zeta. \quad (22)$$

We revert the equation (19) in which the Markov process $\mathbf{U}^*(t)$ 'seeks to model' the temporal fluid particle velocity $\mathbf{U}^L(t)$. For a given initial condition at time t_0, $\mathbf{U}^*(t_0)$ is a Gaussian random variable with zero mean and variance u'^2; for $t > t_0$, $\mathbf{U}^*(t)$ describes the Ornstein-Uhlenbeck (OU) process (the stationary random process; as the Wiener process is nowhere differentiable, the Langevin equation is, in a strict sense, only heuristic). As a constant drift term characterizes the Wiener process, the Ornstein–Uhlenbeck process holds well when it is dependent on the current value of the process. The latter is true in the present case.

The intrinsic inadequacy is that while $\mathbf{U}^L(t)$ is differentiable, $\mathbf{U}^*(t)$ is not. Therefore the model fails a qualitative test if $\mathbf{U}^*(t)$ is examined on an infinitesimal time scale (see Pope [17] and Minier [24]). However, a resulting brilliant analysis [17] using the behaviour of the autocorrelations function shows that much as an uninterestingly negative slope is recorded at the origin slope at very small times ζ/T, owing to $\mathbf{U}^*(t)$ being not differentiable the exponential form provides a very reasonable approximation to the observed autocorrelations for larger times.

4. INTERMITTENCY

Particle trajectories, under normal circumstances, submit to Gaussian statistics. Intermittency arises when deviations from Gaussian statistics get increasingly larger when considered at increasingly smaller scales of fluctuations. It is supposed that the small scales of turbulence are produced due to several interactions down the flow and are, at a reasonably large Reynolds number, independent of the large fluctuation scales and the turbulence production mechanism. The small scales depend uniquely on the mean rate of spectral energy transfer, which under stationary conditions amounts to the mean rate of dissipation of turbulence kinetic energy $\langle \varepsilon \rangle$, and the viscosity ν.

At suitably high Reynolds numbers Sawford and Yeung [30] indicated that the inertial sub-range, which is a portion of the equilibrium range, is detached from both the energy- containing and dissipation scales, and therefore independent of viscosity. It is, in effect, characterized solely by $\langle \varepsilon \rangle$. The dissipation sub-range is the portion of

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the equilibrium range at the smallest scales where the viscosity is essential. Momentous as it is, velocity statistics at various temporal scales may depend on large scale forcing and boundary conditions, as evinced by the concept of *universality*. Admitting the central nature of the concept, Arn`eodo *et al*. [8] investigated intermittency and universality properties of velocity temporal fluctuations obtained from some laboratory data. The measuring of intermittency features of the velocity fluctuations is typically realized through the analysis of the *velocity structure functions*.

Regarding moments of the velocity fluctuations, the Lagrangian Velocity Structure Functions (LVSF) of positive integer order *p* in the form

$$\left\langle |\delta v_i^L(\zeta)|^p \right\rangle = \left\langle |v_i^L(t+\zeta) - v_i^L(t)|^p \right\rangle = G_i^{(p)}(\zeta), \tag{23}$$

where $i = x, y, z$ encode the velocity components along a single particle trajectory, the average is defined over the ensemble of trajectories, and the intermittency expresses itself in the anomalous scaling exponents, λ_p admitting the power law scaling [8, 25]

$$\left\langle |\delta v_i(\zeta)|^p \right\rangle : \zeta^{\lambda_p}. \tag{24}$$

Giving to Kolmogorov (K41)[25] prediction $\lambda_p^{41} = p/3$ with exponents that deviate considerably from the simple scaling prediction, especially for $p > 3$, where $\lambda_p < \lambda_p^{41}$. The scaling exponents λ_p are determined by Taylor's expansion

$$\lambda_p = \sum_{k=1}^{\infty} a_k (-1)^{k+1} p^k / k! \tag{25}$$

As indicated by Arn`eodo *et al.* [8], the decay of the power law is imminent for scales $\zeta \to \zeta_n$, when dissipative effects dominate. The statistics of velocity fluctuations at changing time lag ζ can be quantitatively expressed by the logarithmic derivatives (see [26, 27, 28]) whose Taylor expansion is in the form

$$\log\left\langle |\delta v_i(\zeta)|^p \right\rangle = pA_1(\zeta) + \frac{p^2}{2!} A_2(\zeta) + \frac{p^3}{3!} A_3(\zeta) + ..., \tag{26}$$

where $A_p(\zeta)$ encodes the *p*th-order cumulant of log$|\delta v_i(\zeta)|$. For some derivatives $G_i^{(p)}(\zeta)$ against $G_i^{(2)}(\zeta)$, we get

$$\lambda_i(p,\zeta) = \frac{\mathrm{d}\log G_i^{(p)}(\zeta)}{\mathrm{d}\log G_i^{(2)}(\zeta)} \tag{27}$$

Since $\lambda_i(p,\zeta)$ is ζ-dependence, a scale-by-scale characterization of intermittency may be achieved.

Considering the energy spectrum equivalence, the LVSF, and its higher-order extensions resident in the inertial sub-range have the Kolmogorov similarity applied to Lagrangian statistics defined along fluid particle trajectories through the flow in the form

$$\left\langle |v_i^L(t+\zeta) - v_i^L(t)|^p \right\rangle : \langle \varepsilon \rangle^{p/2} \zeta^{p/2} \qquad (t_\beta = \zeta = T_L) \tag{28}$$

and

$$\left\langle \left| v_i^L(t+\zeta) - v_i^L(t) \right|^p \right\rangle : \; v_\beta^p (\zeta / t_\beta)^p \qquad (\zeta = T_\beta) \tag{29}$$

The dissipation (Kolmogorov) scales are such that length $\beta = (\upsilon^3 / \langle \varepsilon \rangle)^{1/4}$, velocity $v_\beta = (\upsilon \langle \varepsilon \rangle)^{1/4}$ and time $t_\beta = (\upsilon \langle \varepsilon \rangle)^{1/2}$, where υ encodes the viscosity term.

The deviations from Kolmogorov similarity theory are considered a culprit in the event of intermittency. At small scales the dissipation rate of turbulence kinetic energy is not smooth, being an extremely variable quantity. In consequence, the cascade process is not sufficiently described by $\langle \varepsilon \rangle$ alone, and higher moments of the dissipation rate are important **(see** Sawford and Yeung [29]).

In temporal (Lagrangian) terms a specific value of the local dissipation rate is

$$\varepsilon_\zeta(t) = \frac{1}{\zeta} \int_t^{t+\zeta} \varepsilon^L(t') dt' \, , \tag{30}$$

where $\varepsilon^L(t) = \varepsilon(x^L(t), t)$ is the rate of dissipation along a fluid particle trajectory. The point-wise values of the dissipation rate are obtained as $\zeta \to 0$.

The Refined Similarity Hypothesis (RHS) admits writing equations (28) and (29) in the form

$$\left\langle \left| v_i^L(t+\zeta) - v_i^L(t) \right|^p | \, \varepsilon_\zeta \right\rangle : \; \varepsilon_\zeta^{p/2} \zeta^{p/2} \qquad (t_\beta = \zeta = T_L), \tag{31}$$

and

$$\left\langle \left| v_i^L(t+\zeta) - v_i^L(t) \right|^p | \, \varepsilon_\zeta \right\rangle : \; \varepsilon_\zeta^{3p/4} \upsilon^{-p/4} \zeta^p \qquad (\zeta = t_\beta). \tag{32}$$

The dissipation sub-range RSH scaling was tested using (32) for $(\zeta = t_\beta)$ (see [29, 30] for details**)** by using the conditional structure functions. It was seen that for large values of $\varepsilon_\zeta / \langle \varepsilon \rangle$ the conditional structure functions approach the dissipation sub-range RSH scaling but there is a strong departure from RSH scaling as $\varepsilon_\zeta / \langle \varepsilon \rangle \to 0$. As $\zeta \to 0$ the corresponding non-dimensional conditional moments of the acceleration are recovered.

5. CONCLUSION

The most classical problems of Lagrangian turbulence relate to the issue of particle dissipation, with two salient aspects: (a) diffusion of single particles from a point source (typically, Taylor problem) and (b) the relative dispersion of pairs of particles (the Richardson problem). The Lagrangian velocity may be explained by modelling the trajectory of a fluid particle in a multifractal random walk in which long-time correlations and the incidence of large amplitude events at small scales dominate the motion. The associated nature, distribution, and time evolution of the dynamical structures entrenched in the particle flow introduce intermittencies in the flow regime. The LVSF, which submits to the Kolmogorov similarity theory (KST) was applied in studying such intermittencies. Some findings suggest that deviations from the Kolmogorov similarity theory are considered a culprit in the event of intermittency.

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Consent for publication

The authors declare that they consented to the publication of this research work.

References

[1] A. S. Monin, A.M. Yaglom, Statistical Fluid Mechanics, Vol. I, Mechanics of Turbulence. MIT Press, Cambridge, MA., (1975).

[2] N. Mordant, E. Lévêque, J-F. Pinton, Experimental and numerical study of the Lagrangian dynamics of high Reynolds turbulence, New J. Phys. 6 (2004) 116: 1-45, DOI: 10.1088/1367-2630/6/1/116.

[3] N. Mordant, P. Metz, O. Michel, J.F. Pinton, Measurement of Lagrangian velocity in fully developed turbulence, Physical Review Letters (2001)-APS.

[4] S. Ott, J. Mann, An experimental investigation of the relative diffusion of particle pairs in three-dimensional flow, J. Fluid Mech. 422 (2000), 207-223.

[5] A. La Porta, G.A. Voth, A.M. Crawford, J. Alexander, E. Bodenschatz, Fluid particle accelerations in fully developed turbulence. Nature 409 (6823), (2001), 1017-9. DOI: 10.1038/35059027.

[6] L. Biferale, G. Bofetta, A. Celani, B.J. Devinish, A. Lanotte, F. Toschi, Multifractal statistics of Lagrangian and acceleration in turbulence, Phys. Rev. Lett. 93: 064502 (2004).

[7] H. Xu, N. T. Ouellette, E. Bodenschatz, Multifractal dimension of Lagrangian turbulence, Phys. Rev. Lett. 96: 114503 (2006).

[8] R. Friedrich, Statistics of Lagrangian Velocities in Turbulent Flows, Phys. Rev. Lett. 90: 084501 (2003).

[9] A. Arn`eodo, R. Benzi, J. Berg, et al. Universal intermittent properties of particle trajectories in highly turbulent flows, https://arxiv.org/pdf/0802.3776.pdf. (Accessed December 3, 2022).

[10] N. Mordant, J. Delour, E. L´ev^eque, A. Arn´eodo, J-F. Pinton, Long-time correlations in Lagrangian dynamics: A key to intermittency in turbulence, Physical Review Letters, 89 (25) (2002): 254502], DOI: 10.1103/PhysRevLett.89.254502.

[11] S.B. Pope, Turbulent Flows, University Press, Cambridge, (2001).

[12] R. Friedrich, R. C. Hillerbrand, Renormalized perturbation theory for Lagrangian turbulence, [In]: J. Peinke, A. Kittel S. Barth M. Oberlack, Progress in turbulence, Springer, (2005).

[13] C. Tong, Measurements of conserved scalar filtered density function in a turbulent jet, Phys. Fluids 13 (2923), (2001).

[14] P. J. Colucci, F. A. Jaberi, P. Givi, S. B. Pope, Filtered density function for large eddy simulation of turbulent reacting flows, Phys. Fluids 10, 499 (1998).

[15] L. Y. M. Gicquel, P. Givi, Velocity filtered density function for large eddy simulation of turbulent flows, Physics of Fluids 14, 3 (2002), 1196-1213.

[16] M.S.G. Sarnitsky, Inference methods for stochastic turbulence modeling with a closure perspective, A Doctoral dissertation submitted to the Department of Mathematics and Statistics, University of Wyoming, (2021).

[17] S. B. Pope, Lagrangian pdf methods for turbulent flows, Annu. Rev. Fluid Mech. 26 (1994) 23-63.

[18] M. R. H. Sheikhi, T. G. Drozda, P. Givi, and S.B.Pope, Velocity-scalar filtered density function for large eddy simulation of turbulent flows, Physics of Fluids 15, 8 (2003) 2321-2337.

[19] L. Y. M. Gicquel and P. Givia, and F. A. Jaberi, Velocity filtered density function for large eddy Simulation of turbulent flows, Physics of Fluids 14, 3 (2002) 1196-1213.

[20] B. Sawford, F. Guest, Uniqueness and universality of Lagrangian stochastic models of turbulent dispersion, In 8th Symposium on Turbulence and Diffusion, (1988) 96–99.

[21] S. Heinz, Statistical mechanics of turbulent flows. Springer Science & Business Media, (2003).

[22] S. B. Pope, A Lagrangian two-time probability density function equation for inhomogeneous turbulent flows, The Physics of fluids, 26: 2 (1983) 3448–3450.

[23] R.Zamansky Acceleration scaling and stochastic dynamics of a fluid particl in turbulence, https://arxiv.org/pdf/2111.01467.pdf. (Accessed November 18, 2022).

[24] J-P. Minier, some issues and remarks on Lagrangian stochastic models for single-phase turbulence and fibre suspensions, http://calliope.dem.uniud.it›downloads›Minier. (Accessed December 3, 2022).

[25] A. N. Kolmogorov, The local structure of turbulence in Incompressible viscous fluid for very large Reynolds numbers. Doklady Akademii Nauk SSSR, 30 (1941) 301-304.

[26] L. Biferale, E. Bodenschatz, M. Cencini, et al., Lagrangian structure functions in turbulence:A quantitative comparison between experiment and direct numerical simulation, Physics of Fluids 20, 065103 (2008).

[27] I.M. Mazzitelli, D. Lohse, Lagrangian statistics for fluid particles and bubbles in turbulence New J. Phys. 6: 203(2004) 1-29.

[28] E. L´evˆeque, L. Chevillard, J.-F. Pinton, et al., Lagrangian intermittencies in dynamic and static turbulent velocity fields from direct numerical simulations, Journal of Turbulence, 8: 3 (2007) 1-12.

[29] B.L. Sawford ,and P.K. Yeung Turbulent Lagrangian velocity statistics conditioned on extreme values of Dissipation, 2013,Elsevier Ltd. DOI: 10.1016/j.piutam.2013.09.012. (Accessed December 1, 2022).

[30] A.N. Kolmogorov, A refinement of previous hypotheses concerning the local structure of turbulence in a viscous incompressible fluid at high Reynolds number. J Fluid Mech,13 (1962) 82–85.