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Dynamics of Wave function Decay: The Quantum Entanglement and the Measure Process

Piero Chiarelli [1,2]

[1]National Council of Research of Italy, San Cataldo, Moruzzi 1, 56124, Pisa, Italy. Email: pchiare@ifc.cnr.it
[2]Interdepartmental Center "E. Piaggio", Faculty of Engineering, University of Pisa, Diotisalvi 2, 56122, Pisa, Italy.

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ABSTRACT

By employing the stochastic extension of the Madelung quantum-hydrodynamic description within a discrete methodology, we establish a solution using the path integral approach to explore the progression of quantum states' superposition. This investigation aims to understand the eventual establishment of a stable end-state configuration amid the backdrop of gravitational background fluctuations. The model identifies the circumstances that lead to a limited range of interaction for the quantum potential, allowing for the emergence of sizeable, classically described macroscopic phenomena. The theory unveils the lowest achievable level of uncertainty in an open quantum system and investigate its congruence with the localized behaviors of macroscopic classical systems. The study examines agreements and differences with decoherence theory and the Copenhagen interpretation of quantum mechanics, and evaluates the impact of wave function decay on the measurement process. The work also shows that the model is able to explain the deviation of the deuterium state in nuclear physics from the quantum mechanics.

Keywords: Quantum-to-classical transition; Quantum decoherence; Gravitational background; Stochastic Madelung quantum hydrodynamics; Kinetics of wave function decay; Universal self-decoherence; Uncertainty principle and finite speed of light; Self-fluctuating quantum mechanics; Minimum measure duration time.

1. Introduction

The notion that quantum mechanics embodies a stochastic process traces its origins back to Nelson's research [1] and has persisted over time. However, Nelson's assumptions were deemed unsatisfactory due to the imposition of time-inversion symmetry constraints on the stochastic dynamics, limiting its applicability.

A definitive resolution to this matter was provided by Kleinert [2], who utilized the path integral approach to demonstrate that quantum mechanics can be conceptualized as an imaginary-time stochastic process. These imaginary-time quantum fluctuations differ from the more familiar real-time fluctuations, as they give rise to a "reversible" pseudo-diffusion kinetics, expounded by the Madelung quantum hydrodynamic model through the influence of the so-called quantum potential. The quantum pseudo-diffusion is characterized by a diffusion coefficient that lacks a positive definition. The key implication is that processes may transpire over a specific time interval or within a subsection of the system that result in a decrease in entropy. In this study, the author examines the quantum imaginary-time stochastic process within the context of concurrent real-time random noise presence.

The main topics are:

Formulating the equation of motion for the stochastic quantum hydrodynamic system in the presence of fluctuations in spacetime curvature, attributed to the existence of gravitational dark energy.

Developing a path integral solution for the dynamics of quantum systems influenced by stochastic noise, with an emphasis on tracking the progression of quantum states' superposition and their possible relaxation into stable configurations.

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Characterizing the configurations of stationary states under the influence of noise and establishing their relationship with deterministic quantum states.

Defining the specific circumstances in which the deterministic limit of the stochastic theory approaches the quantum mechanics, thereby identifying the conditions for convergence.

Identifying the scenarios in which classical behavior emerges within extensive systems, shedding light on the transition to classical-like phenomena on a larger scale.

Extending the principles of uncertainty relations within fluctuating quantum systems, exploring how these relations are compatible with the presence of stochastic effects.

Investigating the phenomenon of quantum entanglement, analyzing the decay of wave functions, and scrutinizing the intricacies of the measurement process in this stochastic quantum framework.

Comparing the measurement process as described by the stochastic quantum hydrodynamic model with viewpoints from decoherence theory and the Copenhagen interpretation of quantum mechanics.

2. The quantum potential fluctuations elicited by the stochastic gravitational background

The quantum-hydrodynamic representation of the Schrodinger equation

$$-i\hbar \frac{\partial}{\partial t}\psi = \left(\frac{\hbar^2}{2m} \frac{\partial^2}{\partial q_\beta \partial q_\beta} - V_{(q)} \right)\psi \qquad (2.1)$$

for the complex wave function $\psi = |\psi| e^{-\frac{iS}{\hbar}}$, are given [3] by the conservation equation for the mass density $|\psi|^2$

$$\frac{\partial}{\partial t}|\psi|^2 + \frac{\partial}{\partial q_i}(|\psi|^2 \dot{q}_i) = 0 \qquad (2.2)$$

and by the motion equation

$$\ddot{q}_{j(t)} = -\frac{1}{m} \frac{\partial \left(V_{(q)} + V_{qu(|\psi|)} \right)}{\partial q_j} \qquad (2.3)$$

where $\dot{q}_j$ is defined, through the momentum $p_j = \frac{\partial S_{(q,t)}}{\partial q_j}$, where $S_{(q,t)} = -\frac{\hbar}{2} ln \frac{\psi}{\psi^*}$ and where

$$V_{qu} = -\frac{\hbar^2}{2m} \frac{1}{|\psi|} \frac{\partial^2 |\psi|}{\partial q_j \partial q_j}. \qquad (2.4)$$

In order to introduce the metric tensor fluctuations of the space-time background, we assume that:

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The fluctuations of the vacuum curvature (gravitational waves background) leads to mass density fluctuations that are described by the wave function ψ_{vac} with density $|\psi_{vac}|^2$;

The (dark) energy density E of the gravitational background waves is proportional to $|\psi_{vac}|^2$;

The equivalent mass of gravitational background fluctuations m_{vac} is defined by the identity $E = m_{vac}c^2 |\psi_{vac}|^2$

The stochastic gravitational wrinkles originated by the big-bang and spacetime dynamics, contributing to the dark energy background, are approximately assumed to not interact with the physical system (gravitational interaction is sufficiently weak to be disregarded).

In this case the wave function of the overall system ψ_{tot} reads

$$\psi_{tot} \cong \psi\psi_{vac} \qquad (2.5)$$

Moreover, by assuming that, the equivalent mass m_{dark} of the dark energy of gravitational waves is much smaller than the mass of the system (i.e., $m_{tot} = m_{dark} + m \cong m$), the overall quantum potential (2.4) reads

$$\begin{aligned} V_{qu(n_{tot})} &= -\frac{\hbar^2}{2m_{tot}} |\psi|^{-1}|\psi_{vac}|^{-1} \frac{\partial^2 |\psi||\psi_{vac}|}{\partial q_i \partial q_i} \\ &= -\frac{\hbar^2}{2m}\left(|\psi|^{-1} \frac{\partial^2 |\psi|}{\partial q_i \partial q_i} + |\psi_{vac}|^{-1} \frac{\partial^2 |\psi_{vac}|}{\partial q_i \partial q_i} + |\psi|^{-1}|\psi_{vac}|^{-1} \frac{\partial |\psi_{vac}|}{\partial q_i}\frac{\partial |\psi|}{\partial q_i}\right). \end{aligned} \qquad (2.6)$$

Furthermore, given the vacuum mass density fluctuation of wave-length λ

$$|\psi_{vac(\lambda)}|^2 \propto cos^2 \frac{2\pi}{\lambda} q \qquad (2.7)$$

associated to the fluctuation wave-function

$$\psi_{vac} \propto \pm cos \frac{2\pi}{\lambda} q , \qquad (2.8)$$

it follows that the overall fluctuating quantum potential energy read

$$\delta \bar{E}_{qu} = \int_V |\psi_{tot}|^2_{(q,t)} \, \delta V_{qu(q,t)} dV , \qquad (2.9)$$

where

$$\begin{aligned} \delta V_{qu(q,t)} &= -\frac{\hbar^2}{2m}\left(|\psi_{vac}|^{-1} \frac{\partial^2 |\psi_{vac}|}{\partial q_i \partial q_i} + |\psi|^{-1}|\psi_{vac}|^{-1} \frac{\partial |\psi_{vac}|}{\partial q_i}\frac{\partial |\psi|}{\partial q_i}\right) \\ &= \frac{\hbar^2}{2m}\left(\left(\frac{2\pi}{\lambda}\right)^2 + |\psi|^{-1} \frac{\partial |\psi|}{\partial q_i}\left(\pm cos\frac{2\pi}{\lambda}q\right)^{-1}\left(\pm sin\frac{2\pi}{\lambda}q\right)\right) = \frac{\hbar^2}{2m}\left(\left(\frac{2\pi}{\lambda}\right)^2 + |\psi|^{-1} \frac{\partial |\psi|}{\partial q_i} tan\frac{2\pi}{\lambda}q\right) \end{aligned} . \qquad (2.10)$$

For $V \rightarrow \infty$, the unidimensional case leads to

$$\delta\overline{E}_{qu(\lambda)} = \frac{1}{\overline{n}_{tot}V}\frac{\hbar^2}{2m}\int_V n_{tot(q,t)}\left(\left(\frac{2\pi}{\lambda}\right)^2 + |\psi|^{-1}\frac{\partial|\psi|}{\partial q_i}tan\frac{2\pi}{\lambda}q\right)dq$$
$$= \frac{1}{\overline{n}_{tot}V}\frac{\hbar^2}{2m}\left(\left(\frac{2\pi}{\lambda}\right)^2\int_V n_{tot(q,t)}dq + \int_V n_{tot(q,t)}\left(|\psi|^{-1}\frac{\partial|\psi|}{\partial q_i}tan\frac{2\pi}{\lambda}q\right)dq\right)$$
$$\cong \frac{\hbar^2}{2m}\left(\frac{2\pi}{\lambda}\right)^2 \qquad (2.11)$$

In (2.11) it has been used the normalization condition $\int_V |\psi_{tot}|^2_{(q,t)}\, dq = \overline{n}_{tot}V$ and, on large volume ($V \gg \lambda_c^{\ 3}$ see (2.15) below), it has been used the approximation

$$lim_{\lambda\to 0}\int_{-\infty}^{\infty} |\psi_{tot}|^2_{(q,t)}\left(|\psi|^{-1}\frac{\partial|\psi|}{\partial q_i}tan\frac{2\pi}{\lambda}q\right)dq \ll \overline{n}_{tot}V\left(\frac{2\pi}{\lambda}\right)^2. \qquad (2.12)$$

For the three-dimensional case, (2.11) leads to

$$\delta\overline{E}_{qu(\lambda)} \cong \frac{\hbar^2}{2m}\sum_i (k_i)^2 = \frac{\hbar^2}{2m}|k|^2 \qquad (2.13)$$

The result (2.13) shows that the mass/energy density fluctuations, increases as the inverse squared of λ. Being so, the quantum potential fluctuations, of very short-wave length (i.e., $\lambda \to 0$) can lead to unlimited large energy fluctuations even for vanishing noise amplitude $T \to 0$. This fact, in principle, could prevent the realization of the deterministic, zero noise, limit (2.2-4) representing the quantum mechanics, if the background fluctuations would produce a white noise.

Actually, the convergence to the deterministic limit (2.2-4) of quantum mechanics for $T \to 0$ is warranted by the fact that uncorrelated fluctuations on shorter and shorter distances are energetically unlikely so that the noise is not white. Thence, the requirement of convergence to the conventional quantum mechanics for $T \to 0$ is warranted by the special form of the spatial correlation function of the noise as $\lambda \to 0$ [4].

Since each component of spatial frequency $k = \frac{2\pi}{\lambda}$ brings the quantum potential energy contribution (2.11), its probability of happening, reads

$$p_{(\lambda)} \propto exp\left[-\frac{\delta\overline{E}_{qu}}{kT}\right] = exp\left[-\frac{\frac{\hbar^2}{2m}\left(\frac{2\pi}{\lambda}\right)^2}{kT}\right] = exp\left[-\left(\frac{\pi\lambda_c}{\lambda}\right)^2\right] \qquad (2.14)$$

where

$$\lambda_c = \sqrt{2}\frac{\hbar}{(mkT)^{1/2}} \tag{2.15}$$

is the De Broglie length.

The calculation of the correlation function $G(\lambda)$ brings a quite heavy stochastic calculation [4]. A more simple and straight way to obtain $G(\lambda)$ is through the analysis of the spectrum of fluctuations.

From (2.14) the spectrum $S(k)$ of the spatial frequency reads

$$S(k) \propto p(\frac{2\pi}{\lambda}) = exp\left[-\left(\frac{\pi\lambda_c}{\lambda}\right)^2\right] = exp\left[-\left(\frac{k\lambda_c}{2}\right)^2\right] \tag{2.16}$$

From (2.16) we can see that the components with wave-length λ smaller than λ_c go quickly to zero. Besides, from (2.16) the spatial shape $G(\lambda)$ reads

$$G_{(\lambda)} \propto \int_{-\infty}^{+\infty} exp[ik\lambda]S_{(k)}dk \propto \int_{-\infty}^{+\infty} exp[ik\lambda]\, exp\left[-\left(k\frac{\lambda_c}{2}\right)^2\right]dk$$
$$\propto \frac{\pi^{1/2}}{\lambda_c} exp\left[-\left(\frac{\lambda}{\lambda_c}\right)^2\right] \quad . \tag{2.17}$$

The expression (2.17) shows that uncorrelated mass density fluctuations on shorter and shorter distance are progressively suppressed by the quantum potential allowing the realization of the conventional "deterministic" quantum mechanics for systems whose physical length is much smaller than the De Broglie one λ_c.

For the sufficiently general case to be of practical interest, where the mass density noise correlation function can be assumed Gaussian with null correlation time, isotropic into the space and independent among different co-ordinates, it can be assumed of the form

$$<\delta n_{(q_\alpha,t)}, \delta n_{(q_\beta+\lambda,t+\tau)}> = <\delta n_{(q_\alpha)}, \delta n_{(q_\beta)}>_{(T)} \; G(\lambda)\delta(\tau)\delta_{\alpha\beta}, \tag{2.18}$$

where, for system whose physical length $\mathcal{L}$ such as $\frac{\mathcal{L}}{\lambda_c} \ll 1$), reads: $G_{(\lambda)} \cong \frac{1}{\lambda_c} = \frac{1}{\hbar}\sqrt{\frac{mkT}{2}}$.

On this ansatz, equation (2.3) assumes the stochastic form [5] (see appendix A)

$$\ddot{q}_{j(t)} = -\kappa\dot{q}_{j(t)} - \frac{1}{m}\frac{\partial\left(V_{(q)} + V_{qu(\rho)}\right)}{\partial q_j} + \kappa D^{1/2}\xi_{(t)} \tag{2.19}$$

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that, eliminating fast variables, leads to

$$\dot{q} = -\gamma_D \frac{\mathcal{L}^2}{8\alpha\hbar} \frac{\partial \left(V_{(q)} + V_{qu(\rho)} \right)}{\partial q} + \sqrt{\gamma_D} \frac{\mathcal{L}}{2} \sqrt{\frac{kT}{\hbar}} \xi_{(t)} . \qquad (2.20)$$

where the probability mass density function ρ is defined by the Smolukowski conservation equation stemming from (2.19 or 2.20) and obeys to the condition $lim_{T \to 0} \rho = |\psi|^2$ since by (2.17-18) the convergence to the quantum mechanics is warranted.

3. The Langevin-Schrodinger equation from the stochastic quantum hydrodynamic approach

Following the inverse formal process, from the Madelung quantum hydrodynamic representation to the Schrodinger one, the Marcovian process of (2.19) can be represented in the form of the Schrodinger-Langevin equation. In order to show this interesting correspondence, we can generally assume for the stochastic case, the complex field

$$\psi_{(q,t)} = \rho_{(q,t)}{}^{1/2} \, exp \left[i \frac{S_{(q,t)}}{\hbar} \right], \qquad (3.21)$$

where close to the deterministic limit of quantum mechanics we can utilize the identity $\rho_{(q,t)}{}^{1/2} \cong |\psi_{(q,t)}|$, it follows that, by utilizing (2.19), the quantum-hydrodynamic equations lead to

$$\ddot{q}_\alpha = \frac{1}{m}\frac{d}{dt}\frac{\partial S}{\partial q_\alpha} = \frac{\partial}{\partial t}\frac{\partial S}{\partial q_\alpha} + \frac{1}{m}\frac{\partial^2 S}{\partial q_\alpha \partial q_\beta}\frac{\partial S}{\partial q_\beta} = \frac{\partial}{\partial q_\alpha}\left(\frac{\partial S}{\partial t} + \frac{1}{m}\frac{\partial S}{\partial q_\beta}\frac{\partial S}{\partial q_\beta} \right)$$
$$= -\frac{1}{m} \frac{\partial \left(V_{(q)} - \frac{\hbar^2}{2m} \frac{1}{|\psi|} \frac{\partial^2 |\psi|}{\partial q_\beta \partial q_\beta} + \kappa S - q_\beta^{1/2} q_\beta^{1/2} m \kappa D^{1/2} \xi_{(t)} \right)}{\partial q_\alpha} \qquad (3.22)$$

and, finally, to the partial stochastic differential equation

$$m\frac{\partial S}{\partial t} + \frac{\partial S}{\partial q_\beta}\frac{\partial S}{\partial q_\beta} = -\left(V_{(q)} - \frac{\hbar^2}{2m}\frac{1}{|\psi|}\frac{\partial^2 |\psi|}{\partial q_\beta \partial q_\beta} + \kappa S - q_\beta^{1/2} q_\beta^{1/2} m\kappa D^{1/2} \xi_{(t)} + C_{(t)} \right). \qquad (3.23)$$

Equation (3.23) leads to the Langevin-Schrodinger equation, by observing that, for system of physical length $\mathcal{L}$ (such as $\frac{\mathcal{L}}{\lambda_c} \ll 1$), ρ obeys to the Smolukowski conservation equation (see Equations (A.22) in appendix A) that reads

$$\lim_{\frac{\mathcal{L}}{\lambda_c} \to 0} \left(\partial_t \rho_{(q,t)} + \frac{\partial \rho_{(q,t)} \bar{\dot{q}}_i}{\partial q_i} + Q_{diss(q,t)} \right) \cong \left(\partial_t \rho_{(q,t)} + \frac{\partial \rho_{(q,t)} \dot{q}_i}{\partial q_i} + Q_{diss(q,t)} \right) = 0 \qquad (3.24)$$

where $\rho_{(q,t)} = \int \mathcal{N}(q,p,t) d^3 p$ and the diffusional dissipation $Q_{diss(q,t)}$ (see (B.15) in appendix B) reads

$$Q_{diss(q,t)} = \int \begin{pmatrix} 0 \\ \frac{1}{2}\frac{\partial D\mathcal{N}_{(q,p,t)}}{\partial p_\alpha} + \ldots + \frac{1}{n!}\sum_{h=2}^{\infty} \frac{\partial^k C^{(k)}_{\alpha\chi\ldots\ldots\varepsilon}\mathcal{N}_{(q,p,t)}}{\underbrace{\partial p_\chi \ldots \partial p_\varepsilon}_{(k-terms)}} \end{pmatrix} d^{3h} p \quad . \qquad (3.25)$$

In fact, since close to the deterministic limit of quantum mechanics it holds that

$$\lim_{\frac{\lambda_c}{\mathcal{L}}\to\infty} \overline{\dot{q}} = \dot{q} \quad , \qquad (3.26)$$

where upper dash stands for the mean value (see (B24) in appendix B, and

$$\lim_{\frac{\lambda_c}{\mathcal{L}}\to\infty} \rho = |\psi|^2 \quad , \qquad (3.27)$$

it follows that

$$\frac{\partial|\psi|}{\partial t} = -\frac{1}{m}\frac{\partial|\psi|}{\partial q_\alpha}\frac{\partial S}{\partial q_\alpha} - \frac{1}{2m}|\psi|\frac{\partial}{\partial q_\alpha}\frac{\partial S}{\partial q_\alpha} + \frac{Q_{diss(q,t)}}{2|\psi|} \quad . \qquad (3.28)$$

Equation (3.28) with the help of (3.23), leads to the generalized Langevin-Schrodinger equation (GLSE) that for time-independent systems reads

$$-i\hbar\frac{\partial}{\partial t}\psi = \frac{\hbar^2}{2m}\frac{\partial^2}{\partial q_\beta \partial q_\beta}\psi - \left(V_{(q)} + C + \kappa S - q_\beta^{1/2} q_\beta^{1/2} m\kappa D^{1/2}\xi_{(t)} + i\frac{Q_{diss(q,t)}}{2|\psi|^2}\right)\psi \qquad (3.29)$$

Close to the deterministic limit of quantum mechanics (i.e., microscopic system with physical length $\mathcal{L}$ much smaller than λ_c) it is possible to characterize the ability of the system to dissipate by the semiempirical parameter α defined by the relation [5] $\lim_{\frac{\mathcal{L}}{\lambda_c}\to 0} \kappa \cong \lim_{T\to 0} \alpha \frac{2kT}{mD}$. On this ansatz, the realization of the quantum mechanics is warranted (see Equation (2.19)) by the condition $\lim_{\frac{\mathcal{L}}{\lambda_c}\to 0} \alpha = 0$. In this case, it can be readily seen that the GLSE (3.29) reduces to the Schrodinger equation.

4. The quantum Brownian motion

When non-vanishing drag force is present in microscopic systems even if they are very close to the quantum deterministic limit (i.e., $\frac{\mathcal{L}}{\lambda_c} = \varepsilon \ll 1$) (4.30), Equation (3.29) does not converge to the conventional quantum mechanics. When the parameter α remains finite close to quantum limit such as

$$\lim_{\frac{\mathcal{L}}{\lambda_c}\to 0} \alpha = \alpha_0 \quad . \qquad (4.31)$$

Equation (3.29) converges to the quantum Brownian motion. In fact, under condition (4.31) and by utilizing dimensional considerations, the following relations apply:

$$\lim_{\frac{\mathcal{L}}{\lambda_c}\to 0} D = \lim_{T\to 0} \gamma_D \left(\frac{\mathcal{L}}{\lambda_c}\right)^2 \frac{\hbar}{2m} = \lim_{T\to 0} \gamma_D \mathcal{L}^2 \frac{kT}{4\hbar} = 0 \qquad (4.32)$$

$$\lim_{\frac{\mathcal{L}}{\lambda_c}\to 0} \kappa \cong \lim_{T\to 0} \alpha \frac{2kT}{mD} = \alpha_0 \frac{8\hbar}{m\gamma_D \mathcal{L}^2} = finite \quad , \qquad (4.33)$$

$$\lim_{\frac{\mathcal{L}}{\lambda_c}\to 0} Q_{diss(q,t)} = 0 \quad . \qquad (4.34)$$

where γ_D is a pure not-null number.

Thus, by (4.32,34) being $\lim_{\frac{\mathcal{L}}{\lambda_c}\to 0} |\frac{Q_{diss(q,t)}}{2|\psi|^2}| << |\kappa S|$, the term $i\frac{Q_{diss(q,t)}}{|\psi|^2}$ can be disregarded in (3.29) and close to the deterministic limit we have

$$\lim_{T\to 0} i\hbar \frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q_\beta \partial q_\beta}\psi + \left(V_{(q)} + \kappa S - q_\beta^{1/2} q_\beta^{1/2} m\kappa D^{1/2} \xi_{(t)} + C_{(t)}\right)\psi \quad . \qquad (4.35)$$

that describes the quantum Brownian motion.

It is worth noting that Equations (3.29,4.35) can be established by the integrability of the velocity field $\dot{q}_\beta = \frac{1}{m}\frac{\partial S}{\partial q_\beta}$ that can be warranted close to the quantum behavior, but that it may fail in macroscopic large-scale classical system since, generally speaking, the velocity field is not-integrable.

5. The quantum path integral motion equation in presence of stochastic noise

The simplified Markov process (2.20) obeys the Smolukowski integro-differential equation for the Markov probability transition function (PTF) [6]

$$P(q,q_0|t+\tau,t_0) = \int_{-\infty}^{\infty} P(q,z|\tau,t)P(z,q_0|t-t_0,t_0)d^r z \qquad (5.1)$$

where the PTF $P(q,z|\tau,t)$ represents the probability that a quantity of the probability mass density (PMD)

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$\rho(q,t)$ at instant t, in a time interval τ, in a point z, is transferred to the point q [6].

The conservation of the PMD ρ in integral form shows that the PTF generates the displacement of a vector (q,t) – (z,0) according to the rule [6]

$$\rho(q,t) = \int P(q,z\,|\,t,0)\rho_{(z,0)} d^r z \tag{5.2}$$

5.1. Stationary eigenstates in presence of noise

Generally speaking, for the quantum case, equation (5.1) cannot be reduced to a Fokker-Planck equation (FPE), since the quantum potential $V_{qu(\rho)}$ owns a functional dependence by $\rho_{(q,t)}$ and the PTF $P_{(q,q_0\,|\,t+\tau,t_0)}$ is non-Gaussian (see Appendix B) .

Nonetheless, if the initial distribution $\rho_{(q,t_0)}$ is stationary (e.g., quantum eigenstate [7]) and is close to the long-time final stationary distribution ρ_{eq} of the stochastic case, it is possible to assume the approximation

$$V_{qu} \cong -(\frac{\hbar^2}{4m})\left(\left(\frac{\partial^2 \ln \rho_{eq(q)}}{\partial q^2}\right) + \frac{1}{2}\left(\frac{\partial \ln \rho_{eq(q)}}{\partial q}\right)^2\right). \tag{5.3}$$

Being in this case the quantum potential not function of time, the stationary long-time solution (warranted in time independent Hamiltonian potentials by the presence of the viscous force) is given by the Fokker-Plank equation

$$\frac{\partial P_{(q,z|t,0)}}{\partial t} + \frac{\partial P_{(q,z|t,0)}\upsilon}{\partial q} = 0 \tag{5.4}$$

where

$$\upsilon = -\frac{1}{m\kappa}\frac{\partial\left(V_{(q)} - (\frac{\hbar^2}{4m})\left(\left(\frac{\partial^2 \ln \rho_{eq}}{\partial q^2}\right) + \frac{1}{2}\left(\frac{\partial \ln \rho_{eq}}{\partial q}\right)^2\right)\right)}{\partial q} - \frac{D}{2}\frac{\partial \ln \rho_{eq}}{\partial q} \tag{5.5}$$

leading to the final equilibrium ($\upsilon = 0$) identity

$$\frac{1}{m\kappa}\frac{\partial\left(V_{(q)} - (\frac{\hbar^2}{4m})\left(\left(\frac{\partial^2 \ln \rho_{eq}}{\partial q^2}\right) + \frac{1}{2}\left(\frac{\partial \ln \rho_{eq}}{\partial q}\right)^2\right)\right)}{\partial q} + \frac{D}{2}\frac{\partial \ln \rho_{eq}}{\partial q} = 0 \tag{5.6}$$

In appendix C the stationary states of linear systems obeying to (5.6) in presence of small noise are shown. The results show that the quantum eigenstates are stable and maintain their shape (with a small change of their variance) when subject to fluctuations.

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5.2. Evolution of quantum superposition of states submitted to noise

In order to determine the evolution of quantum superposition of states, that are not stationary, (not considering fast kinetics, large fluctuations and jumps) we have to integrate the stochastic differential equation (SDE) (2.19) that eliminating the fast variables reads

$$\dot{q} = -\gamma_D \frac{\mathcal{L}^2}{8\alpha\hbar} \frac{\partial \left(V_{(q)} - (\frac{\hbar^2}{2m}) \left(\left(\frac{\partial^2 \ln \rho}{\partial q^2} \right) + \left(\frac{\partial \ln \rho}{\partial q} \right)^2 \right) \right)}{\partial q} + \sqrt{\gamma_D} \frac{\mathcal{L}}{2} \sqrt{\frac{kT}{\hbar}} \xi_{(t)} \tag{5.7}$$

As shown below, this can be done by using the discrete approach with the help of both the Smolukowski integro-differential equation (5.1) and the associated conservation equation (5.2) for the PMD ρ.

We integrate the SDE (5.7) by using its 2nd order discrete expansion

$$q_{k+1} \cong q_k - \frac{1}{m\kappa} \frac{\partial \left(V_{(q_k)} + V_{qu(\rho_{q_k}, t_k)} \right)}{\partial q_k} \Delta t_k - \frac{1}{m\kappa} \frac{d}{dt} \frac{\partial \left(V_{(q_k)} + V_{qu(\rho_{q_k}, t_k)} \right)}{\partial q_k} \frac{\Delta t_k{}^2}{2} + D^{1/2} \Delta \mathcal{W}_k \tag{5.8}$$

where

$$q_k = q_{(t_k)} \tag{5.9}$$

$$\Delta t_k = t_{k+1} - t_k \tag{5.10}$$

$$\Delta \mathcal{W}_k = \mathcal{W}_{(t_{k+1})} - \mathcal{W}_{(t_k)} \tag{5.11}$$

where $\Delta \mathcal{W}_k$ has Gaussian zero mean and unitary variance whose probability function $\mathcal{P}(\Delta \mathcal{W}_k, \Delta t)$, for $\Delta t_k = \Delta t \ \forall \ k$, reads

$$\begin{aligned} lim_{\Delta t \to 0} \mathcal{P}(\Delta \mathcal{W}_k, \Delta t) &= lim_{\Delta t \to 0} (4\pi D \Delta t)^{-1/2} exp - \frac{\Delta \mathcal{W}_k^2}{4\Delta t} = lim_{\Delta t \to 0} (4\pi D \Delta t)^{-1/2} exp - \frac{1}{4\Delta t} \frac{(q_{k+1} - < q_{k+1} >)^2}{D} \\ &= (4\pi D \Delta t)^{-1/2} exp - \frac{1}{4\Delta t} \frac{\left(q_{k+1} - q_k - < \dot{\bar{q}}_k > \Delta t - \frac{< \ddot{\bar{q}}_k >}{2} \Delta t^2 \right)^2}{D} \end{aligned} \tag{5.12}$$

where it has been introduced the midpoint approximation

$$\bar{q}_k = \frac{q_{k+1} + q_k}{2}, \tag{5.13}$$

where

$$< \dot{\bar{q}}_k > = - \frac{1}{m\kappa} \frac{\partial \left(V_{(\bar{q}_k)} + V_{qu(\rho_{\bar{q}_k}, t_k)} \right)}{\partial \bar{q}_k} \tag{5.14}$$

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$$< \ddot{\overline{q}}_k >= -\frac{1}{2m\kappa}\frac{d}{dt}\frac{\partial\left(V_{(\overline{q}_k)}+V_{qu(\rho_{(\overline{q}_k)},t_k)}\right)}{\partial\overline{q}_k} \qquad (5.15)$$

are the solutions of the deterministic problem:

$$< q_{k+1} >\cong< q_k > -\frac{1}{m\kappa}\frac{\partial\left(V_{(q_k)}+V_{qu(\rho_{q_k},t_k)}\right)}{\partial q_k}\Delta t_k - \frac{1}{m\kappa}\frac{d}{dt}\frac{\partial\left(V_{(q_k)}+V_{qu(\rho_{q_k},t_k)}\right)}{\partial q_k}\frac{\Delta t_k^{\,2}}{2} \qquad (5.16)$$

By using standard manipulations [2], from (5.12), the PTF reads

$$\begin{aligned}
P(q,q_0,t,t_0) &= P(q_n,q_0\,|\,t-t_0,0) = lim_{\Delta t\to 0}\,\mathcal{P}(q_n,q_0\,|\,n\Delta t,0) \\
&= lim_{\Delta t\to 0}\int_{-\infty}^{\infty}\Pi_{k=1}^{n}\,dq_{k-1}\mathcal{P}(q_k,q_{k-1}\,|\,\Delta t,(k-1)\Delta t) \\
&= lim_{\Delta t\to 0}\left\{\Pi_{k=1}^{n}\int_{-\infty}^{\infty}dq_{k-1}\right\}(4\pi D\Delta t)^{-n/2}\exp\left[\begin{matrix}-\frac{1}{2D}\sum_{k=1}^{n}<\dot{\overline{q}}_{k-1}>\Delta q_k-\frac{\Delta t}{4D}\sum_{k=1}^{n}\left(\left(\frac{q_k-q_{k-1}}{\Delta t}\right)^2+<\dot{\overline{q}}_{k-1}>^2\right)\\ -\frac{\Delta t}{4D}\sum_{k=1}^{n}D^{1/2}\Delta\mathcal{W}_k<\ddot{\overline{q}}_{k-1}>_{(q_{k-1})}\end{matrix}\right] \\
&= \int_{q_0}^{q}\mathcal{D}q\,\exp\left[-\frac{1}{2D}\sum_{k=1}^{n}<\dot{\overline{q}}_{k-1}>\Delta q_k\right]\exp-\frac{\Delta t}{4D}\left[\sum_{k=1}^{n}\begin{matrix}\left(\frac{q_k-q_{k-1}}{\Delta t}\right)^2+<\dot{\overline{q}}_{k-1}>^2\\ -2D\frac{\partial<\dot{\overline{q}}_{k-1}>}{\partial\overline{q}_{k-1}}\end{matrix}\right] \\
&= \left(\exp\int_{q_0}^{q}\frac{1}{2D}<\dot{q}>_{(q,t)}dq\right)\int_{q_0}^{q}\mathcal{D}q\,\exp-\frac{1}{4D}\int_{t_0}^{t}dt\left(\dot{q}^2+<\dot{q}>^2+2D\frac{\partial<\dot{q}>}{\partial q}\right)
\end{aligned} \qquad (5.17)$$

where it has been introduced the discrete PTF $\mathcal{P}(q_k,q_{k-1}\,|\,\Delta t,(k-1)\Delta t)$ that reads

$$\begin{aligned}
lim_{\Delta t\to 0}\,\mathcal{P}(q_k,q_{k-1}\,|\,\Delta t,(k-1)\Delta t) &= lim_{\Delta t\to 0}(4\pi D\Delta t)^{-1/2}\exp\frac{\Delta t}{2D}\left[\begin{matrix}-<\dot{\overline{q}}_{k-1}>\left(\frac{q_k-q_{k-1}}{\Delta t}\right)\\ -\frac{1}{2}\left(\begin{matrix}\left(\frac{q_k-q_{k-1}}{\Delta t}\right)^2+<\dot{\overline{q}}_{k-1}>^2\\ -2D\frac{\partial<\dot{\overline{q}}_{k-1}>}{\partial\overline{q}_{k-1}}\end{matrix}\right)\end{matrix}\right] \\
&= lim_{\Delta t\to 0}(4\pi D\Delta t)^{-1/2}\exp\frac{\Delta t}{4D}\left[-\left(\dot{q}_{k-1}-<\dot{\overline{q}}_{k-1}>\right)^2+2D\frac{\partial<\dot{\overline{q}}_{k-1}>}{\partial\overline{q}_{k-1}}\right]
\end{aligned} \qquad (5.18)$$

Since the quantum potential is a function of the PMD ρ

$$\rho(q_k,t_k) = \int_{-\infty}^{\infty}P(q_k,q_0\,|\,t_k,0)\rho(q_0,0)dq_0 , \qquad (5.19)$$

the evolution of equation (5.8) depends on the exact sequence of the noise inputs $D^{1/2}\Delta\mathcal{W}_k$ and, therefore, also on the discrete time interval of integration. This behavior can be easily verified by performing the numerical

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integration of (5.8). The vagueness of the problem can be analytically identified by the fact that, $<\dot{\bar{q}}_{k-1}>_{(<\dot{q}_k>)}$ and $\frac{\partial <\dot{\bar{q}}_{k-1}>_{(<\dot{q}_k>)}}{\partial q_{k-1}}$ depend on $\rho(q_k, k\Delta t)$ and $\rho(q_{k+1}, (k+1)\Delta t)$, that define the quantum potential values $V_{qu(\rho_k)}$, $V_{qu(\rho_{k+1})}$, unknown at the time instant $(k-1)\Delta t$.

Although there is no general solution to this problem, in the limit of small speed $\dot{q}_k$ and small noise amplitude, it is possible to proceed by successive steps of approximation since the existence of the deterministic limit (see appendix E)

$$lim_{\Delta t\to 0}\, lim_{D\to 0}\, \dot{q}_k = lim_{\Delta t\to 0} <\dot{\bar{q}}_k> = <\dot{q}_k> \qquad (5.20)$$

warrants that, for sufficiently short time interval Δt, the speed change is small enough to have that

$$\frac{<\dot{q}_{k+1}>}{<\dot{q}_k>} = 1 + <\varepsilon> \qquad (5.21)$$

with $<\varepsilon> \ll 1$, since $lim_{\Delta t\to 0} <\varepsilon> \to 0$, and that

$$<\dot{\bar{q}}_k> = \frac{<\dot{q}_{k+1}> + <\dot{q}_k>}{2} = <\dot{q}_k>\left(1+\frac{<\varepsilon>}{2}\right) \cong <\dot{q}_k> \qquad (5.22)$$

Therefore, by starting from the zero order of approximation $<\dot{\bar{q}}_k> \cong <\dot{q}_k>$, there exists a sufficiently small noise amplitude, as well as small diffusion coefficient $D = \gamma_D \frac{\mathcal{L}^2}{4}\frac{kT}{\hbar}$, to obtain the PTF by successive steps of approximation where the starting zero-order reads

$$\begin{aligned}
&\mathcal{P}^{(0)}(q_k, q_{k-1} \,|\, \Delta t, (k-1)\Delta t) \\
&= (4\pi D\Delta t)^{-1/2} exp\frac{\Delta t}{4D}\left[-\left(\dot{q}_{k-1} - \frac{<\dot{q}_k> + <\dot{q}_{k-1}>}{2}\right)^2 + D\frac{\partial <\dot{q}_k> + <\dot{q}_{k-1}>}{\partial \bar{q}_{k-1}}\right] \\
&= (4\pi D\Delta t)^{-1/2} exp\frac{\Delta t}{4D}\left[-\left(\dot{q}_{k-1} - <\dot{q}_{k-1}>\left(1+\frac{<\varepsilon>}{2}\right)\right)^2 + D\frac{\partial <\dot{q}_k> + <\dot{q}_{k-1}>}{\partial \bar{q}_{k-1}}\right] \\
&\cong (4\pi D\Delta t)^{-1/2} exp\frac{\Delta t}{4D}\left[-\left(\dot{q}_{k-1} - <\dot{q}_{k-1}>\right)^2 + 2D\frac{\partial <\dot{q}_{k-1}>}{\partial q_{k-1}}\right]
\end{aligned} \qquad (5.23)$$

which can be used to find the zero-order of approximation of the PMD $\rho^{(0)}$ at the next instant k

$$\rho^{(0)}(q_k, k\Delta t) = \int_{-\infty}^{\infty} \mathcal{P}^{(0)}(q_k, q_{k-1} \,|\, \Delta t, (k-1)\Delta t)\, \rho(q_{k-1}, (k-1)\Delta t)\, dq_{k-1} \qquad (5.24)$$

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and to define the approximated quantum potential at the instant k that allows to obtain the expression of the velocity $< \dot{q}_k >^{(0)}$ of the deterministic evolution

$$< \dot{q}_k >^{(0)} = -\frac{1}{m\kappa} \frac{\partial \left(V_{(q_k)} - (\frac{\hbar^2}{4m}) \left(\left(\frac{\partial^2 ln\, \rho^{(0)}{}_{(q_k, t_k)}}{\partial q^2} \right) - \frac{\left(\frac{\partial}{\partial q} ln\, \rho^{(0)}{}_{(q_k, t_k)} \right)^2}{2} \right) \right)}{\partial q_k} . \qquad (5.25)$$

Thence, as shown in appendix D, at successive order of approximation, the final PTF $\mathcal{P}^{(\infty)}(q_k, q_{k-1} \,|\, \Delta t, (k-1)\Delta t)$ reads

$$\mathcal{P}^{(\infty)}(q_k, q_{k-1} \,|\, \Delta t, (k-1)\Delta t) = lim_{u \to \infty} \mathcal{P}^{(u)}(q_k, q_{k-1} \,|\, \Delta t, (k-1)\Delta t)$$
$$\cong \left(4\pi D \Delta t\right)^{-1/2} exp \frac{\Delta t}{4D} \left[-\left(\dot{q}_{k-1} - \frac{< \dot{q}_k >^{(\infty)} + < \dot{q}_{k-1} >}{2} \right)^2 + D \left(\frac{\partial < \dot{q}_k >^{(\infty)}}{\partial q_k} + \frac{\partial < \dot{q}_{k-1} >}{\partial q_{k-1}} \right) \right] . \qquad (5.26)$$

It worth noting that the convergence of (5.26) generally depends by the chaoticity of the classical trajectories of motion of the system, by the amplitude of the noise T and by the discrete time interval Δt .

The existence of the deterministic limit of quantum mechanics warrants the existence of the basin of convergence of (5.26) in the $lim\left(T, \Delta t\right) \to 0$, but its wideness depends by the specificity of each physical system.

If, for discrete values of T and Δt of integration , $\mathcal{P}^{(u)}$ fluctuates and $lim_{u \to \infty} \mathcal{P}^{(u)}$ cannot be precisely determined, the PTF $\mathcal{P}^{(u)}$ can be estimated by taking its mean values beyond the $\bar{u}$-*th* order of approximation such as

$$\mathcal{P}^{(\infty)} \equiv lim_{u \to \infty} Re\, g\left\{ \mathcal{P}^{(u)} \right\} = lim_{u \to \infty} \frac{1}{u - \bar{u} + 1} \sum_{i=\bar{u}}^{u} \mathcal{P}^{(i)} . \qquad (5.27)$$

Finally, by using $\mathcal{P}(q_k, q_{k-1} \,|\, \Delta t, (k-1)\Delta t)$, the PMD at the k -th instant reads

$$\rho^{(\infty)}(q_k, k\Delta t) = \int_{-\infty}^{\infty} \mathcal{P}^{(\infty)}(q_k, q_{k-1} \,|\, \Delta t, (k-1)\Delta t)\, \rho(q_{k-1}, \left(k-1\right)\Delta t)\, dq_{k-1} . \qquad (5.28)$$

leading to the velocity field

$$< \dot{q}_k >^{(\infty)} = -\frac{1}{m\kappa} \frac{\partial \left(V_{(q_k)} - (\frac{\hbar^2}{4m}) \left(\left(\frac{\partial^2 ln\, \rho^{(\infty)}{}_{(q_k, t_k)}}{\partial q^2} \right) - \frac{1}{2} \left(\frac{\partial}{\partial q} ln\, \rho^{(\infty)}{}_{(q_k, t_k)} \right)^2 \right) \right)}{\partial q_k} \qquad (5.29)$$

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As far as it concerns the continuous limit of the PTF, it reads

$$P(q,q_0\,|\,t-t_0,0)=lim_{\Delta t\to 0}\,\mathcal{P}^{(\infty)}(q_n,q_0\,|\,n\Delta t,0)$$

$$=lim_{\Delta t\to 0}\int_{-\infty}^{\infty}\Pi_{k=1}^{n}\,dq_{k-1}\mathcal{P}^{(\infty)}(q_k,q_{k-1}\,|\,\Delta t,(k-1)\Delta t)$$

$$=\int_{q_0}^{q}\mathcal{D}q\,exp\left[-\frac{1}{2D}\sum_{k=1}^{n}<\dot{\bar{q}}_{k-1}>^{(\infty)}\Delta q_k\right]exp-\frac{\Delta t}{4D}\left[\sum_{k=1}^{n}\begin{matrix}\left(\frac{q_k-q_{k-1}}{\Delta t}\right)^2+<\dot{\bar{q}}_{k-1}>^{(\infty)2}\\-2D\frac{\partial<\dot{\bar{q}}_{k-1}>^{(\infty)}}{\partial\bar{q}_{k-1}}\end{matrix}\right]$$

$$=\left(exp\int_{q_0}^{q}\frac{1}{2D}<\dot{q}>_{(q,t)}dq\right)\int_{q_0}^{q}\mathcal{D}q\,exp-\frac{1}{4D}\int_{t_0}^{t}dt\left(\dot{q}^2+<\dot{q}>^2+2D\frac{\partial<\dot{q}>}{\partial q}\right) \quad (5.30)$$

Where it has been used the identity $<\dot{\bar{q}}_{k-1}>^{(\infty)}=\frac{<\dot{q}_k>^{(\infty)}+<\dot{q}_{k-1}>^{(\infty)}}{2}$.

The general solution, given by the recursive formula (5.30), can be applied also to non-linear system that cannot be treated by standard approaches [8].

5.3. General features of relaxation of quantum superposition of states

In the classical case, the FPE describing the Brownian process, admits the stationary long-time solution

$$P(q,q_{-\infty}\,|\,t-t_{-\infty},t_{-\infty})=lim_{t_0\to-\infty}\,N\,exp\frac{1}{D}\int_{q_{-\infty}}^{q}<\dot{q}>_{(q'_{(t,t_0)})}dq'=N\,exp\frac{1}{D}\int_{q_{-\infty}}^{q}K(q')dq' \quad (5.31)$$

where $K(q)=-\frac{1}{m\kappa}\frac{\partial V_{(q)}}{\partial q}$ leading to the canonical expression [2]

$$P(q,q_0\,|\,t-t_0,t_0)=\left(exp\int_{q_0}^{q}\frac{1}{2D}K(q')dq'\right)\int_{q_0}^{q}\mathcal{D}q\,exp-\frac{1}{4D}\int_{t_0}^{t}dt\left(\dot{q}^2+K^2(q)+2D\frac{\partial K(q)}{\partial q}\right) \quad (5.32)$$

Generally speaking, in the quantum case, (5.30) cannot be given in a closed form (5.31) since the quantum potential depends on the specific relaxation path $\rho_{(q,t)}$ of the system toward the steady state which significantly depends on the initial conditions $\rho_{(q,t_0)}$, $<\dot{q}>_{(q,t_0)}$ and, therefore, on the initial time t_0 when the quantum superposition of states are submitted to fluctuations.

Besides, from (5.8) we can see that q_{t_k} depends by the exact sequence of inputs of stochastic noise since the quantum potential is not fixed, but influenced by them. This behavior, in classically chaotic systems, can lead to relevant divergences of the trajectories in a short time. Thus, in principle, different long-period stationary configurations $\rho_{(q,t=\infty)}$ (i.e., eigenstates described by 5.6) can be reached whenever starting from the same superposition of states. Being so, in classically chaotic systems, the Born's rule can be applied also to the measure of the single quantum state.

Even if $\mathcal{L} \ll \lambda_c \cup \lambda_{qu}$, it is noteworthy to observe that, in order to have finite quantum lengths λ_c and λ_{qu} (necessary to have the quantum-stochastic dynamics of Equation (5.7) and the quantum decoupled (classical) environment and/or measuring apparatus) the non-linearity of the system-environment interaction is necessary: The quantum decoherence with the decay of the superposition of states is strongly based upon the ubiquitous classical chaoticity of real systems.

On the other hand, a perfect linear universal system would maintain $\lambda_c = \infty$ as well as quantum correlations on global scale and would never allow the quantum decoupling between the system and the measuring apparatus necessary for the measure process.

Furthermore, since the connection (A.6.28) between the PMD and the MDD holds only at leading order of approximation of $\dot{q}$ (i.e., slow relaxation process and small amplitude of fluctuations), in the case of large fluctuations (that can occur on time scale much longer than the relaxation one) $\rho_{(q,t)(q,t)}$ can make transitions not described by (5.30) even from a stationary eigenstate to a generic superposition of states (e.g., following quantum synchronization [9]). In this case a new relaxation toward different stationary eigenstate will follow: The PMD $\rho_{(q,t)}$ (5.28) describes the relaxation process in the time interval between two large fluctuations, but not the complete evolution of the system toward the statistical mixture. Due to the jumping process on long time scale, the system made by collection of a large number of particles (or independent subsystems) relaxes toward an assigned statistical mixture (whose distribution is determined by the temperature dependence of the diffusion coefficient).

6. Emerging of the classical behavior on large size systems

It's indeed a fact that when one manually nullifies the quantum potential in the quantum hydrodynamic equations (2.1-3), the traditional equation of motion that corresponds to classical mechanics emerges [4]. Even though this might hold true, such an operation lacks mathematical validity as it alters the fundamental properties of the quantum hydrodynamic equations. By taking this step, the stable arrangements (referred to as eigenstates) are eliminated due to the removal of the counterbalancing effect between the quantum potential and the Hamiltonian force [7], which is responsible for establishing the stability of the eigenstates. Therefore, even a minor quantum potential cannot be disregarded within the framework of the deterministic quantum hydrodynamic model. Conversely, in the stochastic generalization it is possible to correctly neglect the quantum potential in (2.19) when its force is much smaller than the force noise ϖ such as $\left| \frac{1}{m} \frac{\partial V_{qu(\rho)}}{\partial q_i} \right| \ll \left| \varpi_{(q,t,T)} \right|$ that by (5.7) leads to

$$\left| \frac{1}{m} \frac{\partial V_{qu(\rho)}}{\partial q_i} \right| \ll \kappa \left(\frac{\mathcal{L}}{\lambda_c} \right) \left(\gamma_D \frac{\hbar}{2m} \right)^{1/2} = \kappa \left(\frac{\mathcal{L}\sqrt{mkT}}{2\hbar} \right) \left(\gamma_D \frac{\hbar}{2m} \right)^{1/2} , \qquad (6.01)$$

and hence, in a coarse-grained description with elemental cell side Δq, to

$$lim_{q\to\Delta q}\left|\frac{\partial V_{qu(n)}}{\partial q_i}\right| \ll m\kappa\left(\frac{\mathcal{L}}{\lambda_c}\right)\left(\frac{\hbar}{2m}\right)^{1/2} = m\kappa\frac{\mathcal{L}}{2}\sqrt{\frac{kT}{2\hbar}}, \qquad (6.02)$$

where $\mathcal{L}$ is the physical system length.

Besides, even if the noise $\varpi_{(q,t,T)}$ has zero mean, the mean of the quantum potential fluctuations $\bar{V}_{st(n,S)} \cong \kappa S$ is not null so that the dissipative force $-\kappa\dot{q}_{(t)}$ in (2.19) appears. In this way, the stochastic sequence of inputs of noise alters the coherent evolution of the quantum superposition of state. Moreover, by observing that the stochastic noise

$$\kappa\left(\frac{\mathcal{L}}{\lambda_c}\right)\left(\gamma_D\frac{\hbar}{2m}\right)^{1/2}\xi_{(t)} \qquad (6.03)$$

grows with the size of the system, it follows that for macroscopic systems (i,e., $\frac{\mathcal{L}}{\lambda_c}\to\infty$), condition (6.01) is satisfied if

$$lim_{\frac{q}{\lambda_c}\to\frac{\mathcal{L}}{\lambda_c}=\infty}\left|\frac{1}{m}\frac{\partial V_{qu(n_{(q)})}}{\partial q_i}\right| < \infty \qquad (6.04)$$

Actually, in order to have a large-scale description, completely free from quantum correlations, we can more strictly require

$$lim_{\frac{q}{\lambda_c}\to\infty}\left|\frac{1}{m}\frac{\partial V_{qu(n_{(q)})}}{\partial q_i}\right| = lim_{\frac{q}{\lambda_c}\to\infty}\frac{1}{m}\sqrt{\frac{\partial V_{qu(n_{(q)})}}{\partial q_i}\frac{\partial V_{qu(n_{(q)})}}{\partial q_i}} = 0. \qquad (6.05)$$

Thus, by observing that for linear systems

$$lim_{q\to\infty} V_{qu(q)} \propto q^2, \qquad (6.06)$$

it immediately follows that they cannot lead to the classical macroscopic phase.

Generally speaking, stronger the Hamiltonian potential higher the wave function localization and larger the quantum potential behavior at infinity [10]. This can be easily proven by observing that given the MDD

$$|\psi|^2 \propto exp\left[-P^k{}_{(q)}\right] \qquad (6.07)$$

where $P^k{}_{(q)}$ is a polynomial of order k, in order to have a finite quantum potential range of interaction, it must result $k < 3/2$, so that linear systems, with $k = 2$, own an infinite range of action of quantum potential.

A concrete illustration can be found in solids that possess a quantum lattice structure. When observing phenomena occurring at intermolecular distances where the interaction follows the linear behavior, quantum characteristics become evident (such as in x-ray diffraction).

However, when focusing on macroscopic attributes (like low-frequency acoustic waves with wavelengths significantly surpassing the linear interatomic distance range), classical behavior becomes predominant.

For instance, for systems that interact by the Lennard-Jones potential, whose long-distance wave function reads [10]

$$lim_{r\to\infty} |\psi| \propto a^{-1/2}\frac{1}{r}, \qquad (6.08)$$

the quantum potential reads

$$lim_{r\to\infty} V_{qu(n)} = lim_{q\to\infty} \frac{\hbar^2}{2m}\frac{1}{|\psi|}\frac{\partial^2|\psi|}{\partial r\partial r} = \frac{1}{r^2} = \frac{\hbar^2}{m}a|\psi|^2 \qquad (6.09)$$

leading to the quantum force

$$lim_{r\to\infty} -\frac{\partial V_{qu(n)}}{\partial r} = lim_{q\to\infty} \frac{\hbar^2}{2m}\frac{\partial}{\partial r}\frac{1}{|\psi|}\frac{\partial^2|\psi|}{\partial r\partial r} = \frac{\hbar^2}{2m}\frac{\partial}{\partial r} r \frac{\partial^2 \frac{1}{r}}{\partial r\partial r} = -2\frac{\hbar^2}{m}\frac{1}{r^3} = 0, \qquad (6.10)$$

so that by (6.01, 6.05), the large-scale classical behavior can appear [10, 11] in a sufficiently rarefied phase.

It is interesting to note that in (6.09) the quantum potential reproduces the hard sphere potential model of the "pseudo potential Hamiltonian model" of the Gross-Pitaevskii equation [12, 13] where $\frac{a}{4\pi}$ is the boson-boson s-wave scattering length.

By observing that, in order to fulfill the condition (6.05) we can sufficiently require that

$$\int_0^\infty r^{-1} \left| \frac{1}{m}\frac{\partial V_{qu(n_{(q)})}}{\partial q_i} \right|_{(r,\theta,\varphi)} dr = limited \qquad \forall \theta,\varphi, \qquad (6.11)$$

so that it is possible to define the quantum potential range of interaction λ_{qu} as [5,10]

$$\lambda_{qu} = \lambda_c \frac{\int_0^\infty r^{-1} \left| \frac{\partial V_{qu(n_{(q)})}}{\partial q_i} \right|_{(r,\theta,\varphi)} dr}{\left| \frac{\partial V_{qu(n_{(q)})}}{\partial q_i} \right|_{(r=\lambda_c,\theta,\varphi)}} = \lambda_c I_{qu} \qquad (6.12)$$

that gives a measure of the physical length of the quantum non-local interactions.

For L-J potentials the convergence of the integral (6.11) for $r \to 0$ is warranted since, at short distance the L-J interaction is linear (i.e., $lim_{r\to 0} V_{qu(r)} \propto r^2$) and

$$lim_{r\to 0}\, r^{-1} \left| \frac{\partial V_{qu(r)}}{\partial r} \right|_{(r,\theta,\varphi)} = constant \quad . \qquad (6.13)$$

6.1. From micro to macro description: the coarse-grained approach

Given the PMD current $J_{j(q,t)} = \rho_{(q,t)} \dot{q}_{j(t)}$, that reads

$$J_{(q,t)} = \rho \dot{q}_{(t)} = -\rho \left(\frac{m\mathcal{L}^2}{4\alpha} \frac{\partial \left(V_{(q)} + V_{qu} \right)}{\partial q} + \varpi_{(q,t,T)} \right)$$

$$= -\rho \left(\frac{1}{m\kappa} \frac{\partial \left(V_{(q)} + V_{qu} \right)}{\partial q_j} + D_{jk}^{1/2} \xi_{k(t)} \right), \qquad (6.14)$$

The macroscopic behavior can be obtained by the discrete coarse-grained spatial description of (6.14), with local cell of side l , that as a function of the j-th cell reads [14]

$$dx_j = -\frac{m\mathcal{L}^2}{4\alpha} \mathcal{D}'_{jm} x_{(m)} \left(\mathcal{D}_{mk} V_k + \mathcal{D}_{mk}^{qu} V_{qu_k} \right) dt + \mathcal{D}''_{jk} x_{(k)} \Phi_{(k)} dW_{k(t)} \qquad (6.15)$$

where

$$x_j = l^3 \rho_{(q_j,t)}, \qquad (6.16)$$

$$V_k = V_{(q_k)}, \qquad (6.17)$$

$$V_{qu_k} = V_{qu(\rho_{(q_k)})}, \qquad (6.18)$$

$$\Phi_k = \Phi_{(q_k,t)}, \qquad (6.19)$$

where

$$lim_{l\to 0}\, l^{-6} < \Phi_j, \Phi_k > = < \varpi_{(q_j)}, \varpi_{(q_k)} >_{(T)} F_{(l(k-j))} . \qquad (6.20)$$

where $F_{(k-j)}$ is the spatial correlation length of the noise, where the terms $\mathcal{D}_{jk}$, $\mathcal{D}'_{jk}$, $\mathcal{D}''_{jk}$ and $\mathcal{D}_{mk}^{qu}$ are matrices of coefficients corresponding to the discrete approximation of the derivatives $\frac{\partial}{\partial q_k}$ at the j-th point.

Generally speaking, the quantum potential interaction V_{qu_k} stemming by the k-th cell, depends by the strength of the Hamiltonian potential $V_{(q_k)}$.

By setting, in a system of a huge number of particles, the side length l equal to the mean intermolecular distance $\mathcal{L}$, and $\mathcal{L}$ is much bigger than the quantum potential length of interaction λ_{qu} we have the realization of the

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classical rarefied phase.

Typically, the Lennard-Jones potential (6.10) leads to:

$$\lim_{\frac{r}{\lambda_{qu}} \to \frac{\mathcal{L}}{\lambda_{qu}} = \infty} -\frac{\partial V_{qu(\rho)}}{\partial r} \approx 2\frac{\hbar^2}{m\lambda_{qu}{}^3}\frac{1}{\left(\frac{r}{\lambda_{qu}}\right)^3} \approx \frac{kT}{\hbar}\sqrt{\frac{mkT}{2}} I_{qu}\frac{1}{\left(\frac{r}{\lambda_{qu}}\right)^3} = 0 \quad , \qquad (6.21)$$

so that the interaction of the quantum potential (stemming by the k-th cell) into the adjacent cells is null and $\mathcal{D}_{mk}^{qu}$ is diagonal. Thus, the quantum effects are confined into each single molecular cell domain.

Furthermore, being for classical systems $\mathcal{L} \gg \lambda_c \cup \lambda_{qu}$, it follows that the spatial correlation length of the noise reads $G_{(k-j)} \propto \delta_{kj}$ and the fluctuations appears spatially uncorrelated in macroscopic classical systems

Conversely, given that for stronger than linearly interacting systems $\lambda_{qu} \to \infty$ so that the quantum potential of each cell extends its interaction to the other ones, the quantum character appears on the coarse-grained large-scale description [10, 15, 16].

6.2. Macroscopic quantum phenomena and transition to the classical behavior

By discretizing the current conservation equation (6.14) for the system of N particles [14], it is possible to obtain the quantum hydrodynamic master equation for macroscopic system of a huge number of molecules.

Generally speaking we observe that, given the range of interaction of the quantum potential λ_{qu}, the De Broglie length λ_c, and the system size $\mathcal{L}$ ($\mathcal{L}^3$ ~represents the mean available volume per molecule in isotropic phase), we can generally distinguish in isotropic systems the cases:

$\mathcal{L} \gg \lambda_{qu}, \lambda_c$

$\mathcal{L} > \lambda_{qu}, \lambda_c$

$\lambda_{qu} > \mathcal{L} > \lambda_c$

$\lambda_c > \mathcal{L}$

In order to describe the typical phases originating by "1-4", we observe that, typically, for L-J potential interacting molecules, the quantum potential range of interaction λ_{qu} extends itself a little bit further the equilibrium position r_0, in the linear zone of interaction, let's say up to $r_0 + \Delta$.

This can be readily checked by assuming the L-J interaction is linear for $r < r_0 + \Delta$, leading to the quantum force

$$-\frac{\partial V_{qu(n)}}{\partial q} \approx \alpha r \quad , \tag{6.22}$$

while for $r > r_0 + \Delta$, by (6.10) we have that

$$-\frac{\partial V_{qu(n)}}{\partial q} \approx 2\frac{\hbar^2}{m}\frac{1}{r^3} . \tag{6.23}$$

On this ansatz λ_{qu} reads

$$\lambda_{qu} \approx \lambda_c \left(\frac{\int_0^{r_0+\Delta} dr}{\lambda_c} + \frac{\int_{r_0+\Delta}^{\infty} \frac{1}{r^4} dr}{\frac{1}{\lambda_c^3}} \right) = r_0 + \Delta + \frac{\lambda_c^4}{3(r_0+\Delta)^3} \tag{6.24}$$

that, for $T \gg 4°k$ (so that for ordinary microscopic mass $m \approx 10^{-27}\,Kg$ we have $\lambda_c \ll r_0 \approx 10^{-9}\,m$ and $\lambda_c \ll \lambda_{qu}$), leads to

$$\lambda_{qu} \approx r_0 + \Delta \quad . \tag{6.25}$$

Thus, for Lennard-Jones interacting particles, under the condition

$$\mathcal{L} \gg \lambda_{qu} \simeq r_0 + \Delta \gg \lambda_c \tag{6.26}$$

of "**Case 1**." we have the rarefied classic gas phases.

Case 2.

The more condensed phase of Lennard-Jones particles, with $\mathcal{L} > \lambda_{qu} \simeq r_0 + \Delta \gg \lambda_c$, still owns a classical behavior since, as a mean, the particles are distant each-other more than the range of interaction of the quantum potential.

In this case, since the inter-particle distance mostly lies in the non-linear range of L-J interaction ($\mathcal{L} > \lambda_{qu} \simeq r_0 + \Delta$) just beyond to the crystalline phase (staring at $\mathcal{L} \cong \lambda_{qu} \simeq r_0 + \Delta$), we typically have a liquid phase [10].

Case 3.

When $\lambda_c > \mathcal{L} > r_0 + \Delta \simeq \lambda_{qu}$ the neighboring molecules lie in the linear intermolecular range of interaction at a distance smaller than the range of non-local quantum potential interaction λ_{qu}.

The observables based on this physical length show quantum behavior (e.g., the Bragg's diffraction of the atomic lattice).

Case 4.

When the temperature is very low ($\lambda_c > \mathcal{L} > r_0 + \Delta \simeq \lambda_{qu}$) and the De Broglie length λ_c becomes so large to overcomes the linear range of interaction (as well as λ_{qu} too), we might have a liquid phase (i.e., $\mathcal{L} > r_0 + \Delta$) showing quantum behavior. This can happen when the intermolecular interaction is so weak to maintain the liquid phase down to very low temperature (e.g., $^4\mathrm{He}$) that allows the De Broglie length to grow up to $\lambda_c > \mathcal{L}$. In this case the observable of fluidity shows the quantum behavior of superfluidity [10,11].

Given the temperature dependence of λ_c and λ_{qu}, we can have quantum-to-classic phase transition in the case 3 and 4, respectively:

when $\lambda_c < \mathcal{L} < \lambda_{qu}$ and $\lambda_{qu} \to \mathcal{L}$, by temperature increase, we can have the solid-fluid transition with melting of crystalline lattice

when $\mathcal{L} < \lambda_c$ and $\lambda_c \to \mathcal{L}$, by temperature increase, we have the superfluid-fluid transition.

Case I.

For a system of Lennard-Jones interacting particles, the quantum potential range of interaction reads

$$\lambda_{qu} \cong \int_0^d dq + \lambda_c{}^4 \int_d^\infty \frac{1}{q^4} dq = d + \frac{\lambda_c}{3}\left(\frac{\lambda_c}{d}\right)^3 \qquad (6.27)$$

where $d = r_0\left(1+\varepsilon\right)$ is the distance up to which the interatomic force is approximately linear ($\varepsilon = \frac{\Delta}{r_0}$) and where r_0 is atomic equilibrium distance.

An experimental confirmation of the physical relevance of quantum potential length of interaction comes from the quantum to classical transition in crystalline solid at melting point when the system passes from a quantum lattice to a fluid amorphous classical phase.

Assuming that, in the quantum lattice, the atomic wave-function (around r_0) spans itself less than the quantum coherence distance, it follows that at the melting point its variance equals $\lambda_{qu} - r_0$.

On these assumptions, the Lindemann constant $L_C = \dfrac{\left\{\begin{matrix} \textit{wave function variance} \\ \textit{at transition} \end{matrix}\right\}}{r_0}$ [10,15] reads $L_C = \dfrac{\lambda_{qu} - r_0}{r_0}$ and it can be theoretically calculated since

$$\lambda_{qu} \approx r_0\left(\left(1+\varepsilon\right) + \frac{1}{3}\left(\frac{\lambda_c / r_0}{\left(1+\varepsilon\right)}\right)^3\right) \approx r_0\left(\left(1+\varepsilon\right) + \frac{1}{3}\left(\frac{\lambda_c}{r_0}\right)^3\right) \qquad (6.28)$$

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that, being $\varepsilon \approx 0,05 \div 0,1$ and $\frac{\lambda_c}{r_0} \approx 0,8$, leads to

$$L_C = \frac{\lambda_{qu} - r_0}{r_0} \approx 0,217 \div 0,267 \quad . \qquad (6.29)$$

More accurate evaluation, making use of the potential well approximation for the molecular interaction [10,11], leads to $\lambda_{qu} \cong 1,2357\ r_0$ and to the value of $L_C = 0,2357$ for the Lindemann constant that well agrees with the measured ones, ranging between 0,2 and 0,25 [15].

Case II.

Since the De Broglie distance λ_c is a function of temperature, the fluid-superfluid transition can be described in monomolecular liquids at very low temperature such as for the $^4\mathrm{He}$. The treatment of this case is detailed in ref. [10,11] where, for the $^4\mathrm{He}$ - $^4\mathrm{He}$ interaction, the potential well is assumed to be

$$V_{(r)} = \infty \qquad 0 < r < \sigma \qquad (6.30)$$

$$V_{(r)} = -0,82\ U \qquad \sigma < r < \sigma + 2\Delta \qquad (6.31)$$

$$V_{(r)} = 0 \qquad \sigma + 2\Delta < r \qquad (6.32)$$

where $U = 10,9\ k_B = 1,5 \times 10^{-22}\ J$ is the Lennard-Jones potential deepness, where $\Delta = 1,54 \times 10^{-10}\ m$ and where $\sigma + \Delta = 3,7 \times 10^{-10}\ m$ is the mean $^4\mathrm{He}$ - $^4\mathrm{He}$ atomic distance.

By posing that at superfluid transition the de Broglie length is of order of the $^4\mathrm{He}$ - $^4\mathrm{He}$ atoms distance so that

$$\sigma < \lambda_c < \sigma + 2\Delta , \qquad (6.33)$$

it follows that for $\lambda_c \approx \sigma$ is about null the ratio of superfluid/normal $^4\mathrm{He}$ density, while for $\lambda_c \approx \sigma + 2\Delta$ we have almost 100% of superfluid $^4\mathrm{He}$. Therefore, at the condition

$$\lambda_c = \sqrt{2}\frac{\hbar}{(mkT_{\lambda_c})^{1/2}} = \sigma + \Delta , \qquad (6.34)$$

when the superfluid/normal $^4\mathrm{He}$ density ratio is at 50%, it follows that the temperature $T_{50\%}$, for the $^4\mathrm{He}$ mass of $m_{^4He} = 6.6 \times 10^{-27}\ kg$, reads

$$T_{50\%} = \frac{2\hbar^2}{mk}\left(\frac{1}{\sigma + \Delta}\right)^2 = \frac{2 \times 1,113 \times 10^{-68}}{6.6 \times 10^{-27} \times 1,38 \times 10^{-23}}\left(\frac{1}{1,3 \times 10^{-19}}\right) = 1,92\ {}^\circ K \qquad (6.35)$$

that well agrees with the experimental data in ref. [16] of about $1,95\ {}^{\circ}K$.

On the other hand, since by (6.33) for $\lambda_c = \sigma + 2\Delta$ all the couples of ^{4}He falls into the quantum state, the superfluid ratio of 100% is reached at the temperature

$$T_{100\%} \approx \frac{2\hbar^2}{mk}\left(\frac{1}{\sigma+2\Delta}\right)^2 = 0,92\ {}^{\circ}K \tag{6.36}$$

well agreeing with the experimental data in ref. [16] of about $1,0\ {}^{\circ}K$.

Moreover, by utilizing the superfluid ratio of 38% at the λ-point of ^{4}He, the transition temperature T_λ reads

$$T_\lambda \approx \frac{2\hbar^2}{mk}\left(\frac{1}{\sigma+0,76\Delta}\right)^2 = 2,20\ {}^{\circ}K \tag{6.37}$$

in good agreement of the measured ^{4}He superfluid transition temperature of $2,17\ {}^{\circ}K$.

It's important to note that the weak nature of Hamiltonian interaction is what paves the way for classical behavior to arise. Indeed, when dealing with systems governed by a quadratic or stronger Hamiltonian potential, the range of interaction attributed to the quantum potential becomes infinite (as seen in equation 6.06), making the attainment of a classical phase unattainable regardless of the system's size [5, 10, 11,15,17].

In this context, the complete expression of classical behavior is exclusively observed on a macroscopic scale within systems that possess sufficiently feeble interactions (weaker than linear and thus classically chaotic). This occurs due to the inability of the quantum potential to extend its non-local influence over vast distances.

Hence, classical mechanics emerges as a decoherent outcome of quantum mechanics in the presence of a fluctuating background metric within the spacetime.

6.3. Measurement process and the finite range of non-local quantum potential interaction

Throughout the process of measurement, the segment of the experimental arrangement responsible for sensing of the system, might experience quantum mechanical interaction. This interaction concludes once the measuring device is moved far away from the system being measured, at a distance significantly greater than λ_c and λ_{qu}. Subsequently, the measuring device handles the interpretation and processing of the 'interaction output.' This usually entails a classical and irreversible procedure that follows a specific direction of time, resulting in the observable outcome of the measurement at a macroscopic scale."

However, decoherence plays a crucial role in the measurement procedure by facilitating the development of a macroscopic classical framework. This framework permits genuine separation between the measurement device and the system on a quantum level, both prior to and after the measurement event. This quantum-disconnected starting and concluding condition is vital for establishing the conclusion of the measurement process and for accumulating a set of statistical data derived from multiple independent measurement repetitions.

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It's worth highlighting that, within the framework of the SQHM, simply taking the measured system to an infinite distance before and after the measurement isn't enough to ensure the separation between the system and the measuring apparatus when $\lambda_c = \infty$ or $\lambda_{qu} = \infty$.

6.4. Minimum measurements uncertainty in quantum systems submitted to stochastic noise

Any quantum theory aiming to depict the development of a physical system across a wide range of sizes must inherently clarify the process through which quantum mechanical traits transition into observable classical conduct on a grander scale. The key differentiating principles between these two explanations are quantum mechanics' minimum uncertainty principle and classical relativistic mechanics' constraint on the finite speed at which interactions and information propagate locally.

If, at a specific distance $\mathcal{L}_q$, which is less than λ_c, a system completely adheres to "deterministic" quantum mechanical progression, causing its individual components to lack separate identities, then for an observer to acquire data regarding the system, the observer must maintain a minimum separation from the observed system (both prior to and subsequent to the procedure) equal or bigger, at least, than the distance $\mathcal{L}_q$. Consequently, due to the finite speed of interaction and information propagation, the procedure cannot be executed in a timeframe briefer than

$$\Delta \tau_{min} > \frac{\mathcal{L}_q}{c} \propto \frac{\lambda_c}{c} \propto \frac{2\hbar}{(2mc^2 kT)^{1/2}}. \quad (6.38)$$

Moreover, given the Gaussian noise (see 2.19, 5.7) (with the diffusion coefficient proportional to kT), we have that the mean value of the energy fluctuation is $\delta E_{(T)} = \frac{kT}{2}$ for degree of freedom. Thence, a non-relativistic ($mc^2 >> kT$) scalar structureless particle of mass m owns an energy variance ΔE

$$\Delta E \approx (<(mc^2 + \delta E_{(T)})^2 - (mc^2)^2>)^{1/2} \cong (<(mc^2)^2 + 2mc^2 \delta E - (mc^2)^2>)^{1/2}$$
$$\cong (2mc^2 < \delta E >)^{1/2} \cong (mc^2 kT)^{1/2} \quad (6.39)$$

from which it follows that

$$\Delta E \Delta t > \Delta E \Delta \tau_{min} \propto \frac{(mc^2 kT)^{1/2} \lambda_c}{c}) \propto \sqrt{2}\hbar, \quad (6.40)$$

It is worth noting that the product $\Delta E \Delta \tau$ is constant since the growing of the energy variance with the square root of T is exactly compensated the equal decrease of the minimum acquisition time τ_{min}.

The same result is achieved if we derive the uncertainty relations between the position and momentum of a particle of mass m.

If we acquire information about the spatial position of a particle with a precision

$$\Delta L > \mathcal{L}_q. \quad (6.41)$$

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the variance Δp of its relativistic momentum $(p^{\mu} p_{\mu})^{1/2} = mc$ due to the fluctuations reads

$$\Delta p \approx (<(mc + \frac{\delta E_{(T)}}{c})^2 - (mc)^2 >)^{1/2} \cong (<(mc)^2 + 2m\delta E - (mc)^2 >)^{1/2}$$
$$\cong (2m < \delta E >)^{1/2} \cong (mkT)^{1/2} \tag{6.42}$$

and the uncertainty relation reads

$$\Delta L \Delta p > \mathcal{L}_q (mkT)^{1/2} \propto \lambda_c (mkT)^{1/2}) \propto \sqrt{2}\hbar \tag{6.43}$$

Equating (6.43) to the uncertainty value such as

$$\Delta L \Delta p > \mathcal{L}_q (2mkT)^{1/2} = \frac{\hbar}{2} \tag{6.44}$$

or

$$\Delta E \Delta t > \Delta E \Delta \tau_{min} = \frac{(2mc^2 kT)^{1/2} \mathcal{L}_q}{c} = \frac{\hbar}{2}, \tag{6.45}$$

it follows that $\mathcal{L}_q = \frac{\lambda_c}{2\sqrt{2}}$, that represents the physical length below which the quantum entanglement is fully effective and represents the minimum (initial and final) distance between the system and the measuring apparatus. As far as it concerns the theoretical minimum uncertainty of quantum mechanics, obtainable from the minimum uncertainty (6.40-45) in the limit of zero noise, we observe that the quantum deterministic behavior (with $\lambda_c \rightarrow \infty$) in the low velocity limit (i.e., $c \rightarrow \infty$) leads to the equalities

$$\Delta \tau_{min} = \frac{\lambda_c}{2c\sqrt{2}} = \frac{\infty}{\infty} \tag{6.46}$$

$$\Delta E \cong (mc^2 kT)^{1/2} = \sqrt{2}\frac{\hbar c}{\lambda_c} = \frac{\infty}{\infty}, \tag{6.47}$$

$$\mathcal{L}_q = \frac{\lambda_c}{2\sqrt{2}} \rightarrow \infty \tag{6.48}$$

$$\Delta p \cong (mkT)^{1/2} \cong \frac{\sqrt{2}\hbar}{\lambda_c} \rightarrow 0 \tag{6.49}$$

but the products

$$lm_{c \rightarrow \infty} \, lim_{\lambda_c \rightarrow \infty} \Delta E \Delta t > \Delta E \Delta \tau_{min} = \frac{\hbar}{2} \tag{6.50}$$

$$lm_{c \rightarrow \infty} \, lim_{\lambda_c \rightarrow \infty} \Delta L \Delta p > \mathcal{L}_q (mkT)^{1/2} = \frac{\hbar}{2} \tag{6.51}$$

remain finite and constitutes the minimum uncertainty of the quantum deterministic limit.

It is interesting to note that in the relativistic limit, due to the finite light speed, the minimum acquisition time of information in the quantum limit reads

$$\Delta \tau_{min} = \frac{\mathcal{L}_q}{c} \to \infty \quad (6.52)$$

The output (6.52) shows that it is not possible to carry out any measurement in the deterministic fully quantum mechanical global system since it is endless.

Moreover, if we want to increase the system spatial precision to $\Delta L' < \mathcal{L}_q < \Delta L$, we can satisfy the condition $\mathcal{L}'_{q(T')} < \Delta L'$ by increasing the temperature to T'.

In this case it follows that minimum uncertainty $\Delta L \Delta p > \mathcal{L}'_{q(T')}(mkT')^{1/2} = \frac{\hbar}{2}$ is unchanged since the it is independent by the temperature. Therefore, the minimum uncertainty relation (6.51) holds whatever the choice of ΔL .

Since non-locality is confined in domains of physical length of order of $\frac{\lambda_c}{2\sqrt{2}}$ and information about a quantum system cannot be transferred faster than the light speed (otherwise also the uncertainty principle is violated) the local realism is established on macroscopic physics while the paradox of the "spooky action at a distance " is limited on microscopic distance (smaller than $\frac{\lambda_c}{2\sqrt{2}}$) where the quantum mechanics fully realize itself.

It must be noted that for the low velocity limit of quantum mechanics the conditions $c \to \infty$ and $\lambda_c \to \infty$ are implicitly assumed into the theory and leads to (apparent) instantaneous transmission of interaction at a distance.

6.5. The stochastic quantum hydrodynamic model and the decoherence theory

In the context of the SQHM, in order to perform statistically reproducible measurement processes and to warrant that the measuring apparatus is fully independent from the measured system (free of quantum potential coupling before and after the measurement), it is necessary to have a global system with a finite length of quantum potential interaction.

In such a case, the SQHM indicates that due to the finite speed of transmission of light and information, it is possible to carry out the measurement within a finite time interval. Therefore, a finite length of quantum potential interaction, and the resulting decoherence, are necessary preconditions for carrying out the measurement process.

The decoherence theory [18-24] does not attempt to explain the problem of measurement and the collapse of the wave function. Instead, it provides an explanation for the transition of the system to the statistical mixture of states generated by quantum entanglement leakage with the environment. Moreover, while the decoherence process may take a long time τ_d for a microscopic system, the decoherence time for macroscopic systems, consisting of n

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microscopic quantum elements, can be very short $\propto \frac{\tau_d}{n}$. However, in the context of the decoherence theory, the superposition of states of the global universal wave function still exists (and remains globally coherent).

This puzzle finds its logical solution in the extensive recurrence time, a concept recently expanded to encompass quantum systems as well [25]. Even within a universally reversible system, certain irreversible phenomena can manifest due to an exceptionally protracted recurrence interval (far surpassing the universe's lifespan). On a certain short time, scale, global quantum systems can imitate classical behaviors so faithfully that distinguishing them from a genuinely classical universe becomes impossible. To illustrate, the timespan required, as calculated by Boltzmann, for a mere cubic centimeter of gas to revert to its initial state involves a staggering number of digits, reaching into the trillions, whereas the age of the universe spans merely thirteen digits.

In the context of Madelung's approach, the Wigner distribution and the quantum hydrodynamic theory are closely connected and do not contradict each other [26]. However, the interpretation of the global system as classical or quantum in nature is ultimately a matter of interpretation. Essentially, we cannot determine whether the noise from the environment is truly random or pseudo-random. In computer simulations, it is widely accepted that any algorithm generating noise will actually produce pseudo-random outputs, but this distinction is not critical in numerical simulations of irreversible phenomena.

The decoherence theory can account for the macroscopic behavior as the result of dissipative quantum dynamics. However, it falls short of specifying the prerequisites essential for establishing a genuinely classical global system. In contrast, the approach of Stochastic Quantum Hydrodynamic Model furnishes a yardstick for identifying the shift from quantum dynamics to classical behavior on a significant macroscopic level. Moreover, the potential, as revealed by SQHM, of attaining a classical global system within a space-time riddled with curvature fluctuations [5] aligns harmoniously with the quantum-gravitational portrayal of the universe in which the gravity is seen as the catalyst for the universal decoherence [27-28].

Furthermore, on a conceptual level, the theory of stochastic quantum Hydrodynamic model tackles the challenging quandary of spontaneous entropy diminishment within the global quantum-reversible system. This entropy reduction is vital for the system to revert to its initial state, as stipulated by the recurrence theorem. Moreover, given that the quantum pseudo-diffusion evolution [29] highlights the co-occurrence of entropic and anti-entropic processes in disparate domains of a quantum system, the puzzle remains unresolved: why haven't we witnessed spontaneous anti-entropic processes occurring somewhere and sometime within the universe?

6.6. The stochastic quantum hydrodynamic theory and the Copenhagen interpretation of quantum mechanics

The path-integral solution of the SQHM (5.26-9) is not general but holds in the small noise limit, before a large fluctuation occurs. It describes the "microscopic stage" of the decoherence process at De Broglie physical length scale. Moreover, the SQHM parametrizes the quantum to classical transition by using two physical lengths, λ_c and λ_{qu}, addressing the quantum mechanics as the asymptotical behavior for $lim\, \lambda_c \to \infty$. Being so, it furnishes additional insight about the measure process.

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Even if the measure process can be treated as a quantum interaction between the system and the measure apparatus, marginal decoherence effects exist for its realization due to:

real decoupling at initial and final state of the measure between the system and the measuring apparatus,

utilization of classical equipment for the experimental management, collection and treatment of the information.

The marginal decoherence is ignored or disregarded because the classical equipment is mistakenly assumed decoupled at infinity, while the assumption of perfect global quantum system (whose interaction extends itself at infinity being $lim\, \lambda_c \to \infty$ and $lim\, \lambda_{qu} \to \infty$) does not allow the realization of such condition.

In order to describe the decoherence during the external interaction V_{ext} , the SQHM reads

$$\dot{p}_j = -m\kappa \dot{q}_{j(t)} - \frac{\partial \left(V + V_{ext} + V_{qu(n)} \right)}{\partial q_j} + m\varpi_{j(q,t,T)} \qquad (6.53)$$

where

$V_{ext} = 0$ for $t < t_0 \cup t > t_0 + \Delta\tau$

In principle, the marginal decoherence, with characteristic time τ_d , may affect the measurement if τ_d is comparable with the measure duration time $\Delta\tau$ (the absence of marginal effects is included in the treatment as the particular case of sufficiently fast measurement with $\frac{\Delta\tau}{\tau_d} \to 0$).

From the general point of view, the SQHM shows that, the steady state after the relaxation depends on its initial configuration

$$|\psi_{(q,t)}> = \sum_n a_n e^{\frac{E_n(t-t_0)}{\hbar}} |\psi_{n(q)}> \qquad (6.54)$$

at the moment t_0 , allowing the system to possibly reach whatever eigenstate of the superposition.

Since the quantum superposition of energy eigenstates possesses a cyclic evolution with recurrence time T, the probability of relaxation to the i-th energy eigenstate for the SQHM model reads

$$P(\psi \to \psi_i) = lim_{N \to \infty} \frac{N_i}{N} \qquad (6.55)$$

where N is the number of time intervals $\Delta t = \frac{\mathrm{T}}{N}$ centered around the time instants t_{0j} (with $0 \leq t_{0j} < \mathrm{T}$ and $j = 1, ..., N$) in which the system is submitted to fluctuations, and N_i is the number of times the i-th energy eigenstate is reached in the final steady state.

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Moreover, since the eigenstates are stable and stationary (see § 5.1) it also follows that the transition probability between the k-th and the i-th ones reads

$$P(\psi_k \to \psi_i) = lim_{N\to\infty} \frac{N_i}{N} = \delta_{ki} \quad (6.56)$$

Since the finite quantum lengths λ_c and λ_{qu} , allowing the quantum decoupling between the system and the measuring apparatus, necessarily implies the "marginal decoherence", it follows that the output of the measure is

produced in a finite time lapse (bigger than $\Delta\tau_{min}$ of (6.38)) due to wave function decay time.

As far as it concerns the Copenhagen interpretation of quantum mechanics, the measurement is a process that produces the wave function collapse and the outcome (e.g., the energy value E_n for the state (6.54)) is described by the transition probability that reads

$$\tilde{P}(\psi \to \psi_i) = \frac{|a_i|^2}{\sum_{j=1}^{j_{max}} |a_j|^2}, \quad (6.57)$$

that for the i-th eigenstate reads

$$\tilde{P}(\psi_k \to \psi_i) = \delta_{ki}, \quad (6.58)$$

In order to analyze the interconnection between the wave function decay and the wave function collapse, we assume, as starting point, that they are different phenomena and have independent realization.

In first instance, we can assume that the wave function decay (with characteristic time τ_d) happens first and then the wave function collapse (with characteristic time τ_c) during the measure process. Without loss of generality we can assume $\Delta\tau_{min} \le \tau_d < \tau_c \ll \Delta\tau$, and thence, in this case it follows that

$$\begin{aligned} lim_{\frac{\tau_d}{\Delta\tau}\to 0} P^{(tot)}(\psi \to \psi_n) &= \sum_i P(\psi \to \psi_i)\tilde{P}(\psi_i \to \psi_n) \\ &= \left(\sum_i lim_{N\to\infty} \frac{N_i}{N} \right) \delta_{in} = lim_{N\to\infty} \frac{N_n}{N} . \end{aligned} \quad (6.59)$$

On the other hand, for the Copenhagen interpretation, the measure on a quantum state with $\Delta\tau \gg \tau_c$, by (6.57-8) it follows that

$$\tilde{P}(\psi \to \psi_n) = \frac{|a_n|^2}{\sum_{k=1}^{t} |a_k|^2} = lim_{\frac{\tau_c}{\Delta\tau}\to 0} P^{(tot)}(\psi \to \psi_n) = lim_{N\to\infty} \frac{N_n}{N} \quad (6.60)$$

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The outputs (6.59-60) show that the wave function collapse, beyond the duration of the wave function decay, is ineffective in the measure. Furthermore, since after the wave function decay the system has already reached its final steady eigenstate, the wave function collapse does not happen since it does not affect the eigenstates.

Being so, we can think to shorten the measure duration $\Delta\tau$ up to the wave function decay time τ_d without have a change in the result (6.59) for the measure, leading the relation

$$\tilde{P}(\psi \rightarrow \psi_n) = \frac{|a_n|^2}{\sum_{k=1}^{t} |a_k|^2} = lim_{N\rightarrow\infty}\frac{N_n}{N} \tag{6.61}$$

On the other side, by considering in second instance that the wave function collapse τ_c is much shorter than the wave function decoherence such as $\Delta\tau_{min} \le \tau_c < \tau_d \ll \Delta\tau$ (i.e., $\frac{\tau_d}{\tau_c} \rightarrow \infty$), the final output reads

$$lim_{\frac{\tau_d}{\Delta\tau}\rightarrow 0} P^{(tot)}(\psi \rightarrow \psi_n) = \tilde{P}(\psi \rightarrow \psi_i)P(\psi_i \rightarrow \psi_n)$$
$$= \left(\sum_i \frac{|a_i|^2}{\sum_{k=1}^{t}|a_k|^2}\right)\left(lim_{N\rightarrow\infty}\frac{N_n}{N}\right) = \sum_i \frac{|a_i|^2}{\sum_{k=1}^{t}|a_k|^2}\delta_{in} = \frac{|a_n|^2}{\sum_{k=1}^{t}|a_k|^2} \quad ,(6.62)$$

showing that the wave function decay, due to the marginal quantum decoherence, does not affect the measure even if it might proceed beyond the wave function collapse. More precisely we can affirm that since the wave function decay does not affect the eigenstates, it does not happen after the wave function collapse.

Therefore, both the wave function collapse as well as the wave function decoherence happens together only during the time of the measure $\Delta\tau > \Delta\tau_{min}$. In the case of (5.101) we might shorten the measure duration time $\Delta\tau$ to τ_c so that, being the wave function decay finished at the end of the measure, identity (6.55,5.101) leads to

$$P^{(tot)}(\psi \rightarrow \psi_n) = \frac{|a_n|^2}{\sum_{k=1}^{t}|a_k|^2} = lim_{N\rightarrow\infty}\frac{N_n}{N} \tag{6.63}$$

The proof of (6.63) can be validated by the solution of the motion equation (5.26-9) of §5.2.

The SQHM through identity (6.63), furnishes the linkage between the wave function collapse and the wave function decay generated by the marginal decoherence possibly showing that they are the same phenomenon.

7. The Stochastic Quantum Hydrodynamics Approach to Hidden Variables puzzle

One of the unresolved facets of quantum theory pertains to the hidden variable approach. While, on the one hand, the hidden variable approach proposed by Bohm [30] falls within the realm of theoretical frameworks, akin to that of Nelson [1], where an attempt is made to replicate the outcomes of quantum mechanics by introducing a form of

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indeterminacy or hidden variable into classical theory, on the other hand, Von Neumann's proof [31] and Bell's theorem [32] unequivocally assert the impossibility of hidden variables replicating quantum mechanics, effectively forbidding the validity of such an approach. However, complicating the situation, Santilli has recently revitalized interest in and the potential validity of the hidden variable hypothesis with his IsoRedShift Mechanics theory [33], which is predicated on introducing a hidden variable to elucidate deviations in nuclear quantum mechanics in the deuterium state. It is worth noting, for precision's sake, that Santilli's hidden variable is integrated into quantum theory but not into classical theory; as such, it can be more accurately regarded as a semi-empirical perturbative parameter rather than a traditional hidden variable. Regarding this matter, the author has recently demonstrated in a paper [34] that the perturbations in quantum mechanical behavior described by the semiempirical parameter introduced by Santilli IsoRedShift Mechanics can actually be attributed to perturbations in quantum mechanics stemming from stochastic fluctuations, a phenomenon well elucidated within the framework of the stochastic generalization of the Madelung quantum representation.

7.1. The stochastic quantum hydrodynamic equation for charged particles with spin

For charged particles with spin the Schrodinger equation [26] leads to the Pauli's equation

$$i\hbar\left(\frac{\partial}{\partial t}-e\phi\right)\Psi=\left(\frac{1}{2m}\left(\frac{\hbar}{i}\frac{\partial}{\partial q_\beta}-eA_\beta\right)^2-\mu B_i\sigma_i\right)\Psi \tag{7.1}$$

where

$$\Psi=\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}=\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix}|\Psi|^2\, e^{i\frac{S}{\hbar}}=\begin{pmatrix}\cos\frac{\vartheta}{2}e^{-i\frac{\varphi}{2}}\\ \sin\frac{\vartheta}{2}e^{i\frac{\varphi}{2}}\end{pmatrix}\rho\, e^{i\frac{S}{\hbar}} \tag{7.2}$$

$$|\Psi|^2=\Psi^\dagger\Psi \tag{7.3}$$

where the probability density of magnetic moment is identified by the vector

$$\mu=\mu\Sigma=\mu\Psi^\dagger\sigma\Psi \tag{7.4}$$

whose versor n reads

$$n=\left(\sin\vartheta\cos\varphi,\sin\vartheta\sin\varphi,\cos\vartheta\right)=\frac{\Psi^\dagger\sigma\Psi}{\Psi^\dagger\Psi}=\frac{\Sigma}{\rho} \tag{7.5}$$

leading to the quantum hydrodynamic equations [26]

$$\frac{\partial}{\partial t}|\Psi|^2+\frac{\partial}{\partial q_i}(|\Psi|^2\,\dot{q}_i\,)=0 \tag{7.6}$$

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$$m\ddot{q}_i = \frac{e}{m}\left(E_i + \varepsilon_{kli}\dot{q}_k B_l\right) - \frac{1}{m}\frac{\partial V_{qu}}{\partial q_i} + \frac{\hbar^2}{4m^2}\frac{1}{|\Psi|^2}\frac{\partial}{\partial q_j}\left(\frac{\partial n_k}{\partial q_i}\right)\left(\frac{\partial n_k}{\partial q_j}\right) \quad (7.7)$$

$$\dot{n}_i = \frac{2\mu}{\hbar}\left(n \times B\right)_i + \frac{\hbar}{2m}\frac{\varepsilon_{ilm}}{|\Psi|^2}\frac{\partial}{\partial q_j}\left(|\Psi|^2 \, n_l \frac{\partial}{\partial q_j} n_m\right) \quad (7.8)$$

where

$$V_{qu} = -\frac{\hbar^2}{2m}\frac{1}{|\Psi|}\frac{\partial^2 |\Psi|}{\partial q_i \partial q_i}, \quad (7.9)$$

For the stochastic problem, the deterministic variables transform into the corresponding probabilistic distributions following the association $|\psi|^2 \to \rho$ and

$$n \to P(n). \quad (7.10)$$

In the deterministic limit when $\delta n \to 0$, since the noise correlation function (2.18-2.19) warrants the convergence to the quantum mechanics, we have that

$$lim_{\ \delta n \to 0} < P(n) >= n \quad (7.11)$$

and, for the stationary eigenstates (i.e., $\dot{n} = 0$ and $\dot{q} = 0$),

$$lim_{\ \delta n \to 0} < P(\dot{n}) > = \dot{n} = 0 \quad (7.12)$$

$$lim_{\ \delta n \to 0} < \dot{q} > = \dot{q} = 0. \quad (7.13)$$

where $< .... >$ stands for the mean value.

Therefore, the difference between the pseudo potential of the deterministic spin motion equation (7.8),

$$\frac{\hbar}{2m}\frac{\varepsilon_{ilm}}{|\Psi|^2}\frac{\partial}{\partial q_j}\left(|\Psi|^2 \, n_l \frac{\partial}{\partial q_j} n_m\right) = \frac{1}{|\Psi|^2}\frac{\partial}{\partial q_j} N_{ij}(|\Psi|^2, n), \quad (7.14)$$

and its stochastic counterpart $\frac{1}{\rho}\frac{\partial}{\partial q_j} N_{ij}(\rho, P(n))$, at first order as a function of $P(\dot{n})$ and $\dot{q}$ (close to the deterministic limit and near stationary condition (7.12)) can be developed in series expansion as

$$\frac{1}{\rho}\frac{\partial}{\partial q_j} N_{ij}(\rho, P(n)) - \frac{1}{|\Psi|^2}\frac{\partial}{\partial q_j} N_{ij}(|\Psi|^2, n)$$
$$\cong C_i + A_{ij}\, P(\dot{n}_j) + B_{ij}\, \dot{q}_{j(t)} \cong C_i + A_{ij}\, \dot{n}_j - \kappa_{s\rho ij}\, \dot{q}_{j(t)}. \quad (7.15)$$

Moreover, by generally posing

$$A_{ij}\, \dot{n}_j = -\overset{\bullet}{\left(\frac{\Psi^{\dagger} \kappa_{ss_{ij}} \sigma_j \Psi}{\Psi^{\dagger} \Psi} \right)}, \qquad (7.16)$$

since from (7.12) $lim_{\ \delta n \to 0} < C >= 0$ so that

$$C_i = \kappa_{s\rho_{ij}} D^{1/2}_{\rho s_j} \xi_{(t)}, \qquad (7.17)$$

the stochastic spin motion equation (7.8) reads

$$\dot{n}_i = \overset{\bullet}{\left(\frac{\Psi^{\dagger} \sigma_i \Psi}{\Psi^{\dagger} \Psi} \right)} = \frac{2\mu}{\hbar}(n \times B) + \frac{\hbar}{2m} \frac{\varepsilon_{ilm}}{|\Psi|^2} \frac{\partial}{\partial q_j}\left(n_l \frac{\partial}{\partial q_j} n_m \right)$$

$$-\overset{\bullet}{\left(\frac{\Psi^{\dagger} \kappa_{ss_{ij}} \sigma_j \Psi}{\Psi^{\dagger} \Psi} \right)} - \kappa_{s\rho_{ij}}\, \dot{q}_{j(t)} + \kappa_{s\rho_{ij}} D^{1/2}_{s\rho_j} \xi_{(t)} \qquad (7.18)$$

Furthermore, if we consider the more simplified situation of stationary mass density distribution with only spin waves noisy oscillations, equation (7.18) reduces to

$$\dot{n}_i = \frac{2\mu}{\hbar}(n \times B) + \frac{\hbar}{2m} \frac{\varepsilon_{ilm}}{|\Psi|^2} \frac{\partial}{\partial q_j}\left(n_l \frac{\partial}{\partial q_j} n_m \right) - \overset{\bullet}{\left(\frac{\Psi^{\dagger} \kappa_{ss_{ij}} \sigma_j \Psi}{\Psi^{\dagger} \Psi} \right)} + \kappa_{s\rho_{ij}} D^{1/2}_{s\rho_j} \xi_{(t)} \qquad (7.19)$$

7.2. The quantum hydrodynamic equation with iso-spin matrices

By utilizing the iso-matrices [33] in the form

$$\hat{\sigma}_i = \sigma_i + \Delta\sigma_i, \qquad (7.20)$$

where

$$\Delta\sigma_1 = \begin{pmatrix} 0 & \lambda - 1 \\ \lambda^{-1} - 1 & 0 \end{pmatrix} \qquad (7.21)$$

$$\Delta\sigma_2 = \begin{pmatrix} 0 & -i(\lambda - 1) \\ i(\lambda^{-1} - 1) & 0 \end{pmatrix} \qquad (7.22)$$

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$$\Delta\sigma_3 = \begin{pmatrix} \lambda^{-1} - 1 & 0 \\ 0 & -(\lambda - 1) \end{pmatrix} \qquad (7.23)$$

the iso-spin versor $\widehat{n}$ of the Madelung quantum hydrodynamic description reads

$$\widehat{n} = \frac{\Psi^\dagger \widehat{\sigma}\Psi}{\Psi^\dagger\Psi} = \frac{\Psi^\dagger \sigma\Psi}{\Psi^\dagger\Psi} + \frac{\Psi^\dagger \Delta\sigma\Psi}{\Psi^\dagger\Psi} = n + \frac{\Psi^\dagger \Delta\sigma\Psi}{\Psi^\dagger\Psi} \qquad (7.24)$$

and the spin motion equation (7.8, 7.14) reads

$$\dot{\widehat{n}} = \frac{2\mu}{\hbar}(n \times B) + \frac{1}{\rho}\partial_j N_{ij} + \left(\frac{\Psi^\dagger \dot{\Delta\sigma}\Psi}{\Psi^\dagger\Psi} \right) \qquad (7.25)$$

By comparing (7.18) with (7.25)

$$\begin{aligned} \dot{\widehat{n}}_i &= \frac{2\mu}{\hbar}(n \times B) + \frac{1}{\rho}\partial_j N_{ij} + \left(\frac{\Psi^\dagger \dot{\Delta\sigma_i}\Psi}{\Psi^\dagger\Psi} \right) = \dot{n}_i \\ &= \frac{2\mu}{\hbar}(n \times B)_i + \frac{1}{\rho}\partial_j N_{ij} - \left(\frac{\Psi^\dagger \dot{\left(\kappa_{(i)ss}\sigma_i\right)}\Psi}{\Psi^\dagger\Psi} \right) + \kappa_{s\rho} D_{s\rho}^{1/2} \xi_{(t)} \end{aligned} \qquad (7.26)$$

we obtain the identity

$$\left(\frac{\Psi^\dagger \dot{\Delta\sigma_i}\Psi}{\Psi^\dagger\Psi} \right) = -\left(\frac{\Psi^\dagger \dot{\kappa_{ss_{ij}}\sigma_j}\Psi}{\Psi^\dagger\Psi} \right) + \kappa_{s\rho} D_{s\rho}^{1/2} \xi_{(t)} \qquad (7.27)$$

That describes how the iso-spin matrices are linked to the stochastic perturbation of the gravitational background.

Moreover, since the fluctuations $\kappa_{s\rho} D_{\rho s}^{1/2} \xi_{(t)}$ due to the gravitational background are quite small respect the drag term $\left(\frac{\Psi^\dagger \dot{\kappa_{ss_{ij}}\sigma_j}\Psi}{\Psi^\dagger\Psi} \right)$ (that is to say that we experience, or measure, a sort of mean value of the hidden variables), it follows that

$$\Delta\sigma_i = -\kappa_{ss_{ij}}\sigma_j , \qquad (7.28)$$

that, for $\lambda = 1 - \varepsilon$ with $\varepsilon \ll 1$, leads to

$$\Delta\sigma_1 = \begin{pmatrix} 0 & \lambda-1 \\ \lambda^{-1}-1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\varepsilon \\ \varepsilon & 0 \end{pmatrix} = -\kappa_{ss1j}\sigma_j = -\kappa_{ss11}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{7.29}$$

$$\Delta\sigma_2 = \begin{pmatrix} 0 & -i\left(\lambda-1\right) \\ i\left(\lambda^{-1}-1\right) & 0 \end{pmatrix} \cong \begin{pmatrix} 0 & i\varepsilon \\ i\varepsilon & 0 \end{pmatrix} = -\kappa_{ss2j}\sigma_j = -\kappa_{ss22}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{7.30}$$

$$\Delta\sigma_3 = \begin{pmatrix} \lambda^{-1}-1 & 0 \\ 0 & -\left(\lambda-1\right) \end{pmatrix} \cong \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix} = -\kappa_{ss3j}\sigma_j = -\kappa_{ss33}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{7.31}$$

and therefore,

$$\kappa_{ss_{ij}} = (1-\lambda)\begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & 0 & 0 \\ 0 & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & 0 \\ 0 & 0 & \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} \tag{7.32}$$

From (7.31) it is worth noting that in the deterministic limit of standard quantum mechanics, for $\lambda = 1$, we have that $\kappa_{(i)ss} = 0$.

In the context of the stochastic quantum hydrodynamic theory, standard quantum mechanics is regarded as a specific scenario that occurs within a static space-time devoid of the fluctuating backdrop of gravitational dark energy.

In the evolving quantum mechanical densities, situated within spacetime featuring a fluctuating gravitational background, the presence of this turbulent dark energy is perceived as a field responsible for inducing a drag force on the oscillations of spin waves and the movement of mass density. This phenomenon exhibits similarities to the effect of the Higgs Boson, where its field imparts inertia to elementary particles as they traverse through it.

It's noteworthy that this gravitational-induced drag force gives rise to quantum decoherence in macroscopic systems and has the potential to manifest classical behavior in systems of considerable size.

8. Conclusion

The stochastic quantum hydrodynamic model introduces a method for elucidating the conduct of quantum systems amidst noise generated by the fluctuating metric of the physical vacuum. According to this model, the noise's spatial spectrum is not white, and its correlation length is endowed with the De Broglie characteristic length. Consequently, the quantum entanglement that remains limited to distances smaller than this characteristic length. However, the range of quantum potential interaction may extend beyond the De Broglie length up to a distance that may be finite in non-linear weakly bonded systems. In this case, as the physical length of the system increases, classical physics may be achieved. Under these circumstances, the quantum potential is not able to maintain the coherence producing the decay of the quantum superposition of states, while leaving eigenstates practically unaffected. The model contains reversible quantum mechanics as the deterministic limit of the theory. The SQHM provides a useful framework for understanding the interplay between quantum mechanics and classical behavior well explaining both the fluid-superfluid transition of ^{4}He and the Lindeman constant at the melting point of crystalline lattice.

The theory shows that in an open quantum system the principle of minimum uncertainty is harmoniously realized if interactions and information do not travel faster than the speed of light.

According to the theory, decoherence is necessary for a quantum measurement to occur in a finite time interval. The theory demonstrates its congruence with the Copenhagen interpretation of quantum mechanics, potentially explaining the wave function collapse as a dynamic phenomenon of wave function decay.

The model reveals, due to the fluctuations of the spacetime background, originating from the Big Bang and by other cosmological source, the quantum evolution of mass densities and spin waves experience a drag force.

The portrayal provided by the SQHM presents a scenario in which classical mechanics emerges on a macroscopic scale within a space-time filled with fluctuations in curvature where gravity functions as the trigger for universal decoherence. The SQHM offers the potential for the following in the future:

i) The study of photon entanglement in a large-scale classical universe. ii) Exploring methods to extend quantum coherence in material systems to achieve stable Q-bits at high temperatures. iii) Simulating quantum nuclear dynamics at extremely high stellar temperatures. iv) Providing the opportunity to develop software for the rapid simulation of Q-bits.

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Conflict of Interest

The author declares that he/she has no conflict of interest.

Consent for Publication

The author declares that he/she consented to the publication of this study.

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Appendix A.

In presence of curvature fluctuations, the mass distribution density (MDD) $|\psi|^2 = \tilde{n}$ becomes a stochastic function that we can ideally pose $\tilde{n} = \overline{n} + \delta n$ where δn is the fluctuating part and $\overline{n}$ is the regular one that obeys to the limit condition

$$lim_{T\to 0}\,\tilde{n} = lim_{T\to 0}\,\overline{n} = |\psi|^2 . \qquad (A.1)$$

The characteristics of the Madelung quantum potential that, in presence of stochastic noise, fluctuates, can be derived by generally posing that is composed by the regular part $\overline{V_{qu(n)}}$ (to be defined) plus the fluctuating one V_{st} such as

$$V_{qu(\tilde{n})} = -\frac{\hbar^2}{2m}\tilde{n}^{-1/2}\frac{\partial^2 \tilde{n}^{1/2}}{\partial q_\beta \partial q_\beta} = \overline{V_{qu(\tilde{n})}} + V_{st} . \qquad (A.2)$$

where the stochastic part of the quantum potential V_{st} leads to the force noise

$$-\frac{\partial V_{st}}{\partial q_i} = m\varpi_{(q,t,T)} . \qquad (A.3)$$

where the noise correlation function reads

$$\lim_{T\to 0} < \varpi_{(q_\alpha,t)},\varpi_{(q_\beta+\lambda,t+\tau)} > = \lim_{T\to 0} < \varpi_{(q_\alpha)},\varpi_{(q_\beta)} >_{(T)} \; G(\lambda)\delta(\tau)\delta_{\alpha\beta}$$
$$\approx \frac{< \varpi_{(q_\alpha)},\varpi_{(q_\beta)} >_{(T)}}{\lambda_c}\delta(\tau)\delta_{\alpha\beta} \qquad (A.4)$$

with

$$\lim_{T\to 0} < \varpi_{(q_\alpha)},\varpi_{(q_\beta)} >_{(T)} = 0 \qquad (A.5)$$

Besides, the regular part $\overline{V_{qu(\tilde{n})}}$, for microscopic systems ($\frac{\mathcal{L}}{\lambda_c} \ll 1$), without loss of generality, can be rearranged as

$$\overline{V_{qu(\tilde{n})}} = -\frac{\hbar^2}{2m}\overline{\left(\tilde{n}^{-1/2}\frac{\partial^2 \tilde{n}^{1/2}}{\partial q_\beta \partial q_\beta}\right)} = -\frac{\hbar^2}{2m}\frac{1}{\rho^{1/2}}\frac{\partial^2 \rho^{1/2}}{\partial q_\beta \partial q_\beta} + \overline{\Delta V} = V_{qu(\rho)} + \overline{\Delta V} \quad \text{(A.6)}$$

where $\rho_{(q,t)}$ is the probability mass density function (PMD) associated to the stochastic process we are going to define that in the deterministic limit obeys to the condition $lim_{T\to 0}\, \rho_{(q,t)} = lim_{T\to 0}\, \tilde{n} = lim_{T\to 0}\, \overline{n} = |\psi|^2$.

Given the quantum hydrodynamic equation of motion (3) for the fluctuating

MDD $\tilde{n} = \overline{n} + \delta n$,

$$\ddot{q}_\alpha = -\frac{1}{m}\frac{\partial\left(V_{(q)} + V_{qu(\tilde{n})}\right)}{\partial q_\alpha} \quad \text{(A.7)}$$

we can rearrange it as

$$\ddot{q}_\alpha = -\frac{1}{m} - \frac{\partial\left(V_{(q)} + V_{qu(\rho)} + \frac{\hbar^2}{2m}\left(\frac{1}{\rho^{1/2}}\frac{\partial^2 \rho^{1/2}}{\partial q_\beta \partial q_\beta} - \frac{1}{\tilde{n}^{1/2}}\frac{\partial^2 \tilde{n}^{1/2}}{\partial q_\beta \partial q_\beta}\right)\right)}{\partial q_\alpha}. \quad \text{(A.8)}$$

The term

$$\frac{\partial}{\partial q_\alpha}\left(\frac{1}{\rho^{1/2}}\frac{\partial^2 \rho^{1/2}}{\partial q_\beta \partial q_\beta} - \frac{1}{\tilde{n}^{1/2}}\frac{\partial^2 \tilde{n}^{1/2}}{\partial q_\beta \partial q_\beta}\right) \quad \text{(A.9)}$$

that in the deterministic case is null since

$$\lim_{\frac{\mathcal{L}}{\lambda_c}\to 0 \text{ or } T\to 0} \tilde{n} = \lim_{\frac{\mathcal{L}}{\lambda_c}\to 0 \text{ or } T\to 0} \left(\overline{n} + \delta n\right) = \overline{n} \equiv \rho, \quad \text{(A.10)}$$

generates an additional acceleration in the motion equation (A.8), which close to the stationary condition (i.e., $\dot{q}=0$), can be developed in the series approximation and reads

$$\frac{\partial}{\partial q_\alpha}\left(\frac{1}{\rho^{1/2}}\frac{\partial^2 \rho^{1/2}}{\partial q_\beta \partial q_\beta} - \frac{1}{\tilde{n}^{1/2}}\frac{\partial^2 \tilde{n}^{1/2}}{\partial q_\beta \partial q_\beta}\right) \cong A_0 + A_1\dot{q} + ... + A_n\dot{q}^n + O(\tfrac{\mathcal{L}}{\lambda_c}). \quad \text{(A.11)}$$

Moreover, since near the limiting condition (A.10) we can pose

$$\lim_{\frac{\mathcal{L}}{\lambda_c}\to 0} \tilde{n}_{(q)} = \lim_{\frac{\mathcal{L}}{\lambda_c}\to 0} \overline{n}_{(q)} + \delta n_{(q,t)} \cong \rho_{(q)} + \varepsilon_{(q)}\delta n_{(t)}, \quad \text{(A.12)}$$

with $\lim_{\frac{\mathcal{L}}{\lambda_c}\to 0} \varepsilon_{(q)} = 1$ and where $\varepsilon_{(q)}$ is smooth with $\frac{\partial^2 \varepsilon^{1/2}}{\partial q_\beta \partial q_\beta}$ finite (since $\lambda_c \gg \mathcal{L} \neq 0$), it follows that

$$lim_{\frac{\mathcal{L}}{\lambda_c}\to 0}\left(\frac{\partial}{\partial q_\alpha}\frac{1}{\rho^{1/2}}\frac{\partial^2\rho^{1/2}}{\partial q_\beta \partial q_\beta}-\frac{\partial}{\partial q_\alpha}\frac{1}{\tilde{n}^{1/2}}\frac{\partial^2\tilde{n}^{1/2}}{\partial q_\beta \partial q_\beta}\right)$$

$$\cong lim_{\frac{\mathcal{L}}{\lambda_c}\to 0}\frac{\partial}{\partial q_\alpha}\left(\frac{1}{\rho^{1/2}}\frac{\partial^2\rho^{1/2}}{\partial q_\beta \partial q_\beta}-\frac{1}{\rho^{1/2}}\frac{\partial^2\rho^{1/2}\left(1+\frac{\varepsilon_{(q)}\delta n_{(t)}}{2\rho}\right)}{\partial q_\beta \partial q_\beta}\right)$$

$$\cong \delta n_{(t)}\frac{\partial}{\partial q_\alpha}\left(-\frac{1}{\rho^{1/2}}\frac{\partial^2\frac{\varepsilon_{(q)}}{2\rho^{1/2}}}{\partial q_\beta \partial q_\beta}\right) \quad , \qquad (A.13)$$

Moreover, given that at the stationary condition (i.e., $<\dot{q}>= lim_{\Delta t\to\infty}\frac{1}{\Delta t}\int_{t-\frac{\Delta t}{2}}^{t\frac{\Delta t}{2}}\dot{q}_{(q,\tau)}d\tau = 0$) it holds:

$$\bar{n}_{(q)(T=0)} = lim_{T\to 0}<\tilde{n}_{(q)}>= lim_{T\to 0}\, lim_{\Delta t\to\infty}\frac{1}{\Delta t}\int_{t-\frac{\Delta t}{2}}^{t\frac{\Delta t}{2}}\tilde{n}_{(q,\tau)}d\tau$$

$$= \tilde{n}_{(q)(T=0)} + lim_{T\to 0}\, lim_{\Delta t\to\infty}\frac{1}{\Delta t}\int_{t-\frac{\Delta t}{2}}^{t\frac{\Delta t}{2}}\delta n_{(t)}d\tau$$

$$\cong \bar{n}_{(q)(T=0)} + lim_{T\to 0}<\delta n_{(t)}>_{(T)} \quad , \qquad (A.14)$$

and thus

$$lim_{T\to 0 or \frac{\mathcal{L}}{\lambda_c}\to 0}<\delta n_{(t)}>_{(T)} = 0 \quad , \qquad (A.15)$$

the mean $< A_0 >$, by (A.13) reads

$$lim_{\frac{\mathcal{L}}{\lambda_c}\to 0}< A_0 >= lim_{\frac{\mathcal{L}}{\lambda_c}\to 0}\, lim_{\dot{q}\to 0}<\frac{\partial}{\partial q_\alpha}\left(\frac{1}{\rho^{1/2}}\frac{\partial^2\rho^{1/2}}{\partial q_\beta \partial q_\beta}-\frac{1}{\tilde{n}^{1/2}}\frac{\partial^2\tilde{n}^{1/2}}{\partial q_\beta \partial q_\beta}\right)>$$

$$\cong lim_{\frac{\mathcal{L}}{\lambda_c}\to 0}<\delta n_{(t)}>\frac{\partial}{\partial q_\alpha}\left(-\frac{1}{\rho^{1/2}}\frac{\partial^2\frac{\varepsilon_{(q)}}{2\rho^{1/2}}}{\partial q_\beta \partial q_\beta}\right) = 0 \quad . \qquad (A.16)$$

Therefore, the general form of the stochastic term A_0 as the zero-mean noise, with null correlation time (see (A.4)), reads

$$A_0 = m\kappa D^{1/2}\xi_{(t)}. \qquad (A.17)$$

Thence, at leading order in $\dot{q}$, sufficiently close to the deterministic limit of quantum mechanics ($\frac{\mathcal{L}}{\lambda_c} \ll 1$), we obtain that

$$-\frac{\hbar^2}{2m}\frac{\partial}{\partial q_\alpha}\left(\frac{1}{\rho^{1/2}}\frac{\partial^2\rho^{1/2}}{\partial q_\beta \partial q_\beta} - \frac{1}{\tilde{n}^{1/2}}\frac{\partial^2\tilde{n}^{1/2}}{\partial q_\beta \partial q_\beta}\right) \cong m\kappa D^{1/2}\xi_{(t)} + A_1\dot{q} + O\left(\left(\frac{\mathcal{L}}{\lambda_c}\right)^2\right)$$

$$\cong \frac{\partial}{\partial q_\alpha}\left(m\kappa D^{1/2}\xi_{(t)} q_\alpha + \kappa S\right) + O\left(\left(\frac{\mathcal{L}}{\lambda_c}\right)^2\right) \qquad (A.18)$$

The first order approximation (A.18) allows to write (A.7) as the Marcovian process

$$\ddot{q}_{j(t)} = -\kappa \dot{q}_{j(t)} - \frac{1}{m}\frac{\partial\left(V_{(q)} + V_{qu(\rho)}\right)}{\partial q_j} + \kappa D^{1/2}\xi_{(t)}. \qquad (A.19)$$

where

$$\rho_{(q,t)} = \int_{-\infty}^{+\infty} \mathcal{N}_{(q,p,t)} d^3 p, \qquad (A.20)$$

where

$$\mathcal{N}(q,\dot{q},t) = \int P(q,\dot{q},z,\dot{z},|\,t,0)\mathcal{N}_{(z,\dot{z},0)} d^3 z d^3 \dot{z} \qquad (A.21)$$

where $P(q,\dot{q},z,\dot{z},|\,t,0)$ is the probability transition function of the Smolukowski conservation equation [6]

$$P(q,p,q_0,p_0|(t'+\tau-t_0),t_0) = \int_{-\infty}^{\infty} P(q,p,q',p'|\,\tau,t')P(q',p',q_0,p_0|\,t'-t_0,t_0)d^3q'd^3p' \qquad (A.22)$$

of the Marcovian process (A.19).

Moreover, by comparing (A.7) in the form

$$\ddot{q}_\alpha = -\frac{1}{m}\frac{\partial\left(V_{(q)} + \overline{V_{qu(\tilde{n})}} + V_{st}\right)}{\partial q_\alpha} = -\frac{1}{m}\frac{\partial\left(V_{(q)} + V_{qu(\rho)} + \overline{\Delta V} + V_{st}\right)}{\partial q_\alpha} \qquad (A.23)$$

with (A.19), it follows that

$$\overline{\Delta V} = \frac{\hbar^2}{2m}\left(\frac{1}{\rho^{1/2}}\frac{\partial^2\rho^{1/2}}{\partial q_\beta \partial q_\beta} - \overline{\left(\tilde{n}^{-1/2}\frac{\partial^2\tilde{n}^{1/2}}{\partial q_\beta \partial q_\beta}\right)}\right) \cong \kappa S \qquad (A.24)$$

and that

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$$V_{st} = q_\beta^{1/2} q_\beta^{1/2} \kappa D^{1/2} \xi_{\alpha(t)} . \qquad \text{(A.25)}$$

Generally speaking, it must be observed that the validity of (A.19) is not general since, as shown in ref. [24-25], friction coefficient $\beta = m\kappa$ is never constant but only in the case of linear harmonic oscillator. Besides, since in order to have the quantum decoupling with the environment, the non-linear interaction is necessary (see relations (6.05-6), actually, the linear case with β constant cannot be rigorously assumed except for the case $\beta = 0$ that corresponds to the deterministic limit of the theory, namely, the conventional quantum mechanics.

Appendix B. The Marcovian noise approximation in presence of the quantum potential

Once the infinitesimal gravitational dark energy fluctuations have broken the quantum coherence on cosmological scale (e.g., for baryonic particles with mass $m \sim 10^{-(30\div 27)} Kg$, it is enough $T \gg 10^{-40} \frac{4\hbar^2}{mk} \sim 10^{-100} {}^\circ K$ in order to have $\lambda_c \ll 10^{20} m$) it follows that the resulting universe can acquire the classical behaviour and it can be divided in quantum decoupled sub-parts (in weak gravity regions with low curvature since the Newtonian gravity is sufficiently feeble for satisfying condition (6.02)). In this context we can postulate the existence of the classical environment.

Thus, for a mesoscale quantum system in contact with a classical environment, it is possible to consider the Markovian process (2.19)

In presence of the quantum potential, the evolution of the MDD $\tilde{n}_{(q,t)}$, from the initial configuration, determined by (5.8) depends by the exact random sequence of the force inputs of the Marcovian noise.

On the other hand, the probabilistic phase space mass density $\mathcal{N}_{(q,p,t)}$ of the Smolukowski equation

$$P(q,p,q_0,p_0 | (t'+\tau-t_0),t_0) = \int_{-\infty}^{\infty} P(q,p,q',p' | \tau,t') P(q',p',q_0,p_0 | t'-t_0,t_0) d^3q' d^3p' \qquad \text{(B.1)}$$

for the Marcovian process (2.19) [6] owns particular properties given by the presence of the quantum potential.

By using the method due to Pontryagin [6] the Smolukowski equation can be transformed into the differential conservation equation for the PTF $P(q,p,t) = P(q,p,q_0,p_0 | t,0)$

$$\frac{\partial P(q,p,t)}{\partial t} + \frac{\partial P(q,p,t)\mathcal{V}_i}{\partial x_i} , \qquad \text{(B.2)}$$

where $x_i = (q_i, p_i)$. Moreover, in the classical case (i.e., $V_{qu(\rho)} = 0$), the Gaussian character of the PTF is warranted by the property that the cumulants higher than two [6]

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$$C^{(n)}_{im_1........m_n} = lim_{\tau\to 0}\frac{1}{\tau}\int\left(y_i - x_i\right)\left(y_{m_1} - x_{m_1}\right)\ldots\left(y_{m_n} - x_{m_n}\right)P(q,p,t)d^{3h}y \qquad \text{(B.3)}$$

are null in the current

$$P(q,p,t)\mathcal{V}_i = -P(q,p,t)\frac{1}{\beta}\frac{\partial\left(V_{(q)} + V_{qu(\rho(q,q_0|t+\tau,t_0))}\right)}{\partial x_i} - \frac{1}{2}\frac{\partial D_{im}P(q,p,t)}{\partial x_m} + .. + \frac{1}{n!}\sum_{n=2}^{\infty}\frac{\partial^n C^{(n)}_{im_1........m_n}P_{(q,z|t,0)}}{\partial x_{m_1}\ldots\partial x_{m_n}} \qquad \text{(B.4)}$$

This condition is satisfied in classical systems since the continuity of the Hamiltonian potential leads to velocities that remain finite as $\tau \to 0$ leading to non-zero contribution just for first term $\frac{(y_i - q_i)}{\tau}$ in (B.3).

In the quantum case, since the quantum potential depends by the derivatives of $\tilde{n}_{(q,t)} = \overline{n}_{(q,t)} + \delta n \cong |\psi|^2 + \delta n$ that is continuous but not derivable, in the limit of $\tau \to 0$ cumulants higher than two can be different from zero and contribute to the probability transition function $P(y,x|\tau,t)$.

Given the conservation equation (B.2) that also holds for the phase space density $\mathcal{N}(q,p,t)$

$$\frac{\partial\mathcal{N}(q,p,t)}{\partial t} + \frac{\partial\mathcal{N}(q,p,t)\mathcal{V}_i}{\partial x_i} = \frac{\partial\mathcal{N}_{(q,p,t)}}{\partial t} + \frac{\partial\mathcal{N}_{(q,p,t)}\bar{\dot{x}}_i}{\partial x_i}$$
$$= \frac{\partial\mathcal{N}}{\partial t} - \frac{\partial\mathcal{N}\bar{\dot{q}}_\alpha}{\partial q_\alpha} - \frac{\partial\mathcal{N}\bar{\dot{p}}_\beta}{\partial p_\beta} + \frac{\partial\left(\frac{1}{2}\frac{\partial C^{(1)}_{im}\mathcal{N}_{(q,p,t)}}{\partial x_m} + \ldots + \frac{1}{n!}\sum_{n=2}^{\infty}\frac{\partial^n C^{(n)}_{im........l}\mathcal{N}_{(q,p,t)}}{\partial x_m\ldots\partial x_l}\right)}{\partial x_i} = 0 \qquad \text{(B.5)}$$

$$\mathcal{V}_i = \begin{pmatrix}\dot{\mathcal{Q}}\\ \dot{\mathcal{P}}\end{pmatrix} = \bar{\dot{x}}_i - \frac{1}{2P_{(x,z|t,0)}}\frac{\partial D_{im}P_{(x,z|t,0)}}{\partial x_m} + \ldots + \frac{1}{n!\,P_{(x,z|t,0)}}\sum_{h=2}^{\infty}\frac{\partial^k C^{(k)}_{im........l}P_{(x,z|t,0)}}{\underbrace{\partial x_m\ldots\partial x_l}_{(2k-terms)}} \qquad \text{(B.6)}$$

$$\bar{\dot{x}}_i = \begin{pmatrix}\bar{\dot{q}}_\beta\\ \bar{\dot{p}} = m\bar{\ddot{q}}_{\alpha(t)}\end{pmatrix} = \begin{pmatrix}\dot{q}_\beta\\ -\frac{\partial\left(V_{(q)} + V_{qu}\right)}{\partial q_\alpha}\end{pmatrix} \qquad \text{(B.7)}$$

$$\dot{x}_i = \begin{pmatrix}\dot{q}_\beta\\ \dot{p} = m\ddot{q}_{\alpha(t)}\end{pmatrix} = \begin{pmatrix}p_\beta / m\\ -\frac{\partial\left(V_{(q)} + V_{qu}\right)}{\partial q_\alpha} + D_p{}^{1/2}\xi_{\alpha(t)}\end{pmatrix} \qquad \text{(B.8)}$$

where

$$V_{qu} = -\frac{\hbar^2}{2m}\frac{1}{\rho^{1/2}}\frac{\partial^2 \rho^{1/2}}{\partial q_\gamma \partial q_\gamma}, \qquad \text{(B.9)}$$

$$\rho_{(q,t)} = \int \mathcal{N}(q,p,t)d^{3h}p, \qquad \text{(B.10)}$$

by integrating it over the momenta, we obtain

$$\frac{\partial \int \mathcal{N} d^{3h}p}{\partial t} - \frac{\partial \int \mathcal{N}\dot{q}_\alpha d^{3h}p}{\partial q_\alpha} - <\dot{p}>_{(q,t)\beta} \int_{-\infty}^{\infty} \frac{\partial \mathcal{N}}{\partial p_\beta} d^{3h}p$$
$$+\int \frac{\partial\left(\frac{1}{2}\frac{\partial C^{(1)}_{im}\mathcal{N}_{(q,p,t)}}{\partial x_m} + \dots + \frac{1}{n!}\sum_{n=2}^{\infty}\frac{\partial^n C^{(n)}_{im\dots\dots\dots l}\mathcal{N}_{(q,p,t)}}{\underbrace{\partial x_m \dots \partial x_l}_{n-terms}}\right)}{\partial x_i} d^{3h}p = 0 \qquad \text{(B.11)}$$

that, with the condition $lim_{p\to\infty} \mathcal{N}_{(q,p,t)} = 0$ and by posing

$$\overline{\dot{q}}_\alpha = \frac{\int \mathcal{N}\dot{q}_\alpha d^{3h}p}{\int \mathcal{N} d^{3h}p}, \qquad \text{(B.12)}$$

leads to

$$\frac{\partial n}{\partial t} - \frac{\partial n\overline{\dot{q}}_\alpha}{\partial q_\alpha} = -\int \frac{\partial\left(\frac{1}{2}\frac{\partial C^{(1)}_{im}\mathcal{N}_{(q,p,t)}}{\partial x_m} + \dots + \frac{1}{n!}\sum_{n=2}^{\infty}\frac{\partial^n C^{(n)}_{im\dots\dots\dots l}\mathcal{N}_{(q,p,t)}}{\underbrace{\partial x_m \dots \partial x_l}_{n-terms}}\right)}{\partial x_i} d^{3h}p \qquad \text{(B.13)}$$

where $C^{(1)}_{im} = p\begin{pmatrix} 0 & 0 \\ 0 & \tilde{D}\delta_{\alpha\beta} \end{pmatrix}$ so that (B.5) can be rearranged as

$$\frac{\partial \rho}{\partial t} = \overline{\dot{q}}_\alpha \frac{\partial \rho}{\partial q_\alpha} + \rho\frac{\partial \overline{\dot{q}}_\alpha}{\partial q_\alpha} + Q_{diss(q,t)} \qquad \text{(B.14)}$$

where

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$$Q_{diss(q,t)} = \int \begin{pmatrix} 0 \\ \dfrac{\partial \left(\dfrac{1}{2} \dfrac{\partial \tilde{D} \delta_{\alpha\chi} \mathcal{N}_{(q,p,t)}}{\partial p_\chi} + \ldots + \dfrac{1}{n!} \sum_{h=2}^{\infty} \dfrac{\partial^k C^{(k)}_{\alpha\chi\ldots\ldots\ldots\varepsilon} \mathcal{N}_{(q,p,t)}}{\underbrace{\partial p_\chi \ldots \partial p_\varepsilon}_{(k-terms)}} \right)}{\partial p_\alpha} \end{pmatrix} d^{3h} p$$

$$= \int \frac{\partial}{\partial p_\alpha} \begin{pmatrix} 0 \\ \dfrac{1}{2} \dfrac{\partial \tilde{D} \mathcal{N}_{(q,p,t)}}{\partial p_\alpha} + \ldots + \dfrac{1}{n!} \sum_{h=2}^{\infty} \dfrac{\partial^k C^{(k)}_{\alpha\chi\ldots\ldots\ldots\varepsilon} \mathcal{N}_{(q,p,t)}}{\underbrace{\partial p_\chi \ldots \partial p_\varepsilon}_{(k-terms)}} \end{pmatrix} d^{3h} p \quad \text{(B.15)}$$

describes the compressibility of the mass density distribution as a consequence of dissipation.

Appendix C. Harmonic Oscillator Eigenstates in fluctuating spacetime

In the case of linear systems

$$V_{(q)} = \frac{m\omega^2}{2} q^2 , \quad \text{(C.1)}$$

the equilibrium condition, referring to the stationary configuration of the eigenstates, leads to

$$\frac{\hbar^2}{4m} \frac{\partial \left(\left(\dfrac{\partial^2 ln n}{\partial q^2} \right) + \dfrac{1}{2} \left(\dfrac{\partial ln n}{\partial q} \right)^2 \right)}{\partial q} = m\kappa D \frac{\partial ln n}{\partial q} + m\omega^2 q \quad \text{(C.2)}$$

that is satisfied by the solution $n = n_0 \, exp\left[-\frac{q^2}{\Delta q^2} \right]$, where $n_{0(q)}$ is defined by the relation

$$\frac{\hbar^2}{4m^2} \frac{\partial \left(\left(\dfrac{\partial^2}{\partial q^2} \left(ln n_0 - \dfrac{q^2}{\Delta q^2} \right) \right) + \dfrac{1}{2} \left(\dfrac{\partial}{\partial q} \left(ln n_0 - \dfrac{q^2}{\Delta q^2} \right) \right)^2 \right)}{\partial q} = \kappa D \frac{\partial \left(ln n_0 - \dfrac{q^2}{\Delta q^2} \right)}{\partial q} + \omega^2 q \, . \quad \text{(C.3)}$$

For the fundamental eigenstate

$$lim_{T \to 0} \, n_0^{1/2} \, exp\left[-\frac{q^2}{2\Delta q^2} \right] = \left(\frac{1}{4\pi} \frac{m\omega}{\hbar} \right)^{1/4} exp\left[-\frac{q^2}{2\Delta q_0^{\,2}} \right] \quad \text{(C.4)}$$

from (C.3), it follows that

$$\frac{1}{\Delta q^2} = -\frac{m^2}{\hbar^2}\kappa D + \frac{m\omega}{\hbar}\sqrt{1+\left(\frac{m}{\hbar\omega}\kappa D\right)^2} \qquad \text{(C.5)}$$

that close to the quantum mechanical state (i.e., $D \to 0$, $\kappa \to 0$ with $\omega \gg \frac{m}{\hbar}\kappa D$) leads to

$$\frac{1}{\Delta q^2} \cong \frac{m\omega}{\hbar} - \frac{m^2}{\hbar^2}\kappa D + \frac{1}{2}\frac{m^3}{\hbar^3\omega}(\kappa D)^2 \cong \frac{m\omega}{\hbar} - \frac{m^2}{\hbar^2}\kappa D \cong \frac{m\omega}{\hbar}\left(1-2\alpha\frac{kT}{\hbar\omega}\right). \qquad \text{(C.6)}$$

and to the distribution

$$n_{eq} = n_0\, exp\left[-\frac{m\omega}{\hbar}\left(1-2\alpha\frac{kT}{\hbar\omega}\right)q^2\right] \qquad \text{(C.7)}$$

where

$$n_0 = \left[\frac{1}{4\pi}\frac{m\omega}{\hbar}\left(1-2\alpha\frac{kT}{\hbar\omega}\right)\right]^{1/2}. \qquad \text{(C.8)}$$

From (C.7-8) it can be observed that, within the limit of small fluctuations, the mass density distribution of the fundamental eigenstate does not lose its Gaussian form but gains a small increase of its variance following the law

$$\Delta q^2 = \Delta q_0^2\left(1+2\alpha\frac{kT}{\hbar\omega}\right). \qquad \text{(C.9)}$$

where

$$\Delta q_0{}^2 = \left(\frac{\hbar}{m\omega}\right). \qquad \text{(C.10)}$$

Moreover, in presence of fluctuation, the energy E_0 of the fundamental stationary state given by (C.11)

$$\begin{aligned} E_0 =< \psi_0 \,|\, H \,|\, \psi_0 >= & \int_{-\infty}^{\infty} n_{(q,t)}\left[\frac{m}{2}\dot{q}^2 + \frac{m\omega^2}{2}q^2 + V_{qu}\right]dq = \int_{-\infty}^{\infty} n_{(q,t)}\left[\frac{m\omega^2}{2}q^2 + V_{qu}\right]dq \\ & = \int_{-\infty}^{\infty} n_{(q,t)}\left[\frac{m\omega^2}{2}q^2 - \frac{m\omega^2}{2}\left(1-2\alpha\frac{kT}{\hbar\omega}\right)q^2 + (n+\frac{1}{2})\hbar\omega\right]dq = \int_{-\infty}^{\infty} n_{(q,t)}\left[\alpha\frac{\omega mkT}{\hbar}q^2 + \frac{\hbar\omega}{2}\right]dq \\ & = \frac{\hbar\omega}{2} + \alpha\frac{\omega mkT}{\hbar}\int_{-\infty}^{\infty} q^2\left(\frac{m\omega}{\pi\hbar}\right)^{1/2} exp\left[-\left(\frac{m\omega}{\hbar} - \frac{2\alpha mkT}{\hbar^2}\right)q^2\right]dq \qquad , \qquad \text{(C.11)} \\ & = \frac{\hbar\omega}{2} + \alpha\frac{\omega mkT}{\hbar}\left(\frac{1}{1+2\alpha\frac{kT}{\hbar\omega}}\right)\Delta q^2 \cong \frac{\hbar\omega}{2} + \alpha kT \end{aligned}$$

shows an energy increases proportional to kT through the dissipation parameter α.

As far as it concerns the energy variance of the fundamental stationary state

$$\Delta E_0 =<\psi_0 | (E-E_0)^2 | \psi_0 >^{1/2}, \quad (C.12)$$

in presence of fluctuation, it reads

$$\begin{aligned}(\Delta E_0)^2 &= \int_{-\infty}^{\infty} n_{(q,t)} \left[\left(\frac{m}{2}\dot{q}^2 + \frac{m\omega^2}{2} q^2 + V_{qu} \right) - \left(\frac{\hbar\omega}{2} + \alpha kT \right) \right]^2 dq \\ &= \int_{-\infty}^{\infty} n_{(q,t)} \left[\alpha \frac{\omega mkT}{\hbar} q^2 - \alpha kT \right]^2 dq \\ &= (\alpha kT)^2 \int_{-\infty}^{\infty} n_{(q,t)} \left[\frac{\omega m}{\hbar} q^2 - 1 \right]^2 dq = \alpha kT \left(-1 + \frac{\omega m}{\hbar} \int_{-\infty}^{\infty} n_{(q,t)} \left(2q^2 + \frac{\omega m}{\hbar} q^4 \right) dq \right) \\ &\cong (\alpha kT)^2 \left(-1 + 1 + \left(\frac{\omega m}{\hbar} \right)^2 \int_{-\infty}^{\infty} n_{(q,t)} q^4 dq \right) \cong \frac{3}{8} (\alpha kT)^2 \end{aligned} \quad (C.13)$$

that allows to measure the dissipation parameter α by the formula

$$\alpha \cong \sqrt{\frac{8}{3}} \frac{\Delta E_0}{kT} . \quad (C.14)$$

For higher eigenstates the eigenvalues of the Hamiltonian read

$$E_j \cong (j + \frac{1}{2})\hbar\omega - (1 + \Delta_j) \alpha kT \quad (C.15)$$

where

$$\Delta_j = \frac{\Delta q^2}{(\Delta q_j^2 - \Delta q_{j-1}^2)} - 1 \quad (C.16)$$

where Δq_j^2 is the wave function variance of the j-th eigenstate and Δq^2 is the variance of the fundamental one.

It is noteworthy to see that the parameter α can be also experimentally evaluated by the measure of the energy gap $\Delta E_j = E_j - E_{j-1}$ between eigenstates through the relation

$$\alpha \cong \frac{\Delta q^2}{(\Delta q_j^2 - \Delta q_{j-1}^2)} \frac{\hbar\omega - (E_{j+1} - E_j)}{kT} \quad (C.17)$$

Appendix D.

At the next order of approximation, the PTF and the associated PMD (5.23-5.25) read, respectively,

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$$\mathcal{P}^{(1)}(q_k,q_{k-1}|\Delta t,(k-1)\Delta t)$$
$$=\left(4\pi D\Delta t\right)^{-1/2} exp\frac{\Delta t}{4D}\left[-\left(\dot{q}_{k-1}-\frac{<\dot{q}_k>^{(0)}+<\dot{q}_{k-1}>}{2}\right)^2+D\frac{\partial<\dot{q}_k>^{(0)}+<\dot{q}_{k-1}>}{\partial q_{k-1}}\right]$$
$$\cong\left(4\pi D\Delta t\right)^{-1/2} exp\frac{\Delta t}{4D}\left[-\left(\dot{q}_{k-1}-\frac{<\dot{q}_k>^{(0)}+<\dot{q}_{k-1}>}{2}\right)^2+2D\left(\frac{\partial<\dot{q}_k>^{(0)}}{\partial q_k}+\frac{\partial<\dot{q}_{k-1}>}{\partial q_{k-1}}\right)\right] \quad (D.1)$$

and

$$\rho^{(1)}(q_k,k\Delta t)=\int_{-\infty}^{\infty}\mathcal{P}^{(1)}(q_k,q_{k-1}|\Delta t,(k-1)\Delta t)\,\rho(q_{k-1},(k-1)\Delta t)dq_k \quad (D.2)$$

that leads to the mean velocity

$$<\dot{q}_k>^{(1)}=-\frac{1}{m\kappa}\frac{\partial\left(V_{(q_k)}-(\frac{\hbar^2}{4m})\left(\left(\frac{\partial^2 ln\rho^{(1)}{}_{(q_k,t_k)}}{\partial q^2}\right)-\frac{1}{2}\left(\frac{\partial}{\partial q}ln\rho^{(1)}{}_{(q_k,t_k)}\right)^2\right)\right)}{\partial q_k}. \quad (D.3)$$

Thence, repeating the procedure, at successive u-th order of approximation (u=2, ,3, ...r) we obtain

$$\mathcal{P}^{(u)}(q_k,q_{k-1}|\Delta t,(k-1)\Delta t)$$
$$=\left(4\pi D\Delta t\right)^{-1/2} exp\frac{\Delta t}{4D}\left[-\left(\dot{q}_{k-1}-\frac{<\dot{q}_k>^{(u-1)}+<\dot{q}_{k-1}>}{2}\right)^2+D\frac{\partial<\dot{q}_k>^{(u-1)}+<\dot{q}_{k-1}>}{\partial q_{k-1}}\right]$$
$$\cong\left(4\pi D\Delta t\right)^{-1/2} exp\frac{\Delta t}{4D}\left[-\left(\dot{q}_{k-1}-\frac{<\dot{q}_k>^{(u-1)}+<\dot{q}_{k-1}>}{2}\right)^2+D\left(\frac{\partial<\dot{q}_k>^{(u-1)}}{\partial q_k}+\frac{\partial<\dot{q}_{k-1}>}{\partial q_{k-1}}\right)\right] \quad (D.4)$$

and

$$\rho^{(u)}(q_k,k\Delta t)=\int_{-\infty}^{\infty}\mathcal{P}^{(u)}(q_k,q_{k-1}|\Delta t,(k-1)\Delta t)\,\rho(q_{k-1},(k-1)\Delta t)dq_k \quad (D.5)$$

$$<\dot{q}_k>^{(u)}=-\frac{1}{m\kappa}\frac{\partial\left(V_{(q_k)}-(\frac{\hbar^2}{4m})\left(\left(\frac{\partial^2 ln\rho^{(u)}{}_{(q_k,t_k)}}{\partial q^2}\right)-\frac{1}{2}\left(\frac{\partial}{\partial q}ln\rho^{(u)}{}_{(q_k,t_k)}\right)^2\right)\right)}{\partial q_k}, \quad (D.6)$$

Appendix E. Convergence to the deterministic continuous limit

The condition $lim_{\Delta t \to 0} lim_{D \to 0} \varepsilon \to 0$ is warranted by the existence of the deterministic continuous limit that implies that

$$\begin{aligned}
& lim_{D\to 0} \mathcal{P}(q_k , q_{k-1} \,|\, \Delta t, (k-1)\Delta t) \\
&= lim_{D\to 0} lim_{\Delta t\to 0} \left(4\pi D\Delta t\right)^{-1/2} exp \frac{1}{4D}\left[-\left(\dot{q}_{k-1} - <\dot{\bar{q}}_{k-1}>\right)^2 \Delta t + 2D \frac{\partial <\dot{\bar{q}}_{k-1}>}{\partial \bar{q}_{k-1}} \Delta t \right] \\
&= lim_{\Delta t\to 0} lim_{D\to 0} \left(4\pi D\Delta t\right)^{-1/2} exp \frac{1}{4D}\left[-\left(\dot{q}_{k-1} - <\dot{\bar{q}}_{k-1}>\right)^2 \Delta t + 2D \frac{\partial <\dot{q}_{k-1}>}{\partial \bar{q}_{k-1}} \Delta t \right] \\
&= lim_{\Delta t\to 0} \Delta t^{-1/2} \delta\left(\left(\dot{q}_{k-1} - <\dot{\bar{q}}_{k-1}> \right)^2 \Delta t \right) = \delta\left(lim_{\Delta t\to 0} \left(\dot{q}_{k-1} - <\dot{\bar{q}}_{k-1}> \right)^2 \right)
\end{aligned} \quad , \qquad \text{(E.1)}$$

and that

$$lim_{\Delta t \to 0} lim_{D \to 0} \dot{q}_{k-1} = lim_{\Delta t \to 0} <\dot{\bar{q}}_{k-1}> = <\dot{q}_{k-1}> . \qquad \text{(E.2)}$$